Analysis of nonlinear pilot-vehicle systems
using modern control theory

by

GARTEUR Action Group FM(AG12)

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Summary

This report ‘Analysis of nonlinear pilot-vehicle systems using modern control theory’ presents a demonstration of modern analysis techniques applied to Pilot Induced Oscillations (PIO) problems. The document is a deliverable of Workpackage 1 (task 1.1) 'Analysis Challenge' of GARTEUR action group FM(AG12) 'Pilot-in-the-Loop Oscillations; Analysis and Test Techniques for their Prevention'.

Four analysis methods have been considered: Bifurcation, Mu, Integral Quadratic Constraints (IQC), and Robust Stability. The document contains tutorials of all methods and describes how they are used to predict Category II PIO for which nonlinear rate limiting and position limiting are central factors. For each method, the setup of a framework suitable for analysis is detailed, the results of the experimentation on one or both of two sets of existing handling qualities data bases (FOSIM data base and SAAB data base) are presented. The aim of this analysis challenge was to evaluate the potentialities of the proposed approaches with respect to well-established criteria such as the Open Loop Onset Point (OLOP) criterion developed at DLR.

In general, the results were found to be promising, new criteria have been formulated, even though a better correlation with the existing criteria and PIO rating scales needs to be done.

The next step of the research will be the application of the proposed methodologies, together with the OLOP criterion, to ADMIRE a full combat aircraft model developed by FFA. The analysis will be supported by new experimental data from flight simulations. This report was established with contributions from CIRA, UNAP, DLR, DUT and ONERA. Integration and final editing of the supported material was performed by DUT.
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Mathematical quantities and symbols

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<td>Parameters nominal values vector</td>
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<td>Adimensional amplitude limit</td>
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## Abbreviations

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<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>ADMIRe</td>
<td>Aero-Data Model in Research Environment</td>
</tr>
<tr>
<td>EREA</td>
<td>Association of European Research Establishments in Aeronautics</td>
</tr>
<tr>
<td>AG</td>
<td>Action Group</td>
</tr>
<tr>
<td>APC</td>
<td>Aircraft-Pilot Coupling</td>
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<td>APR</td>
<td>Average Phase Rate</td>
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<tr>
<td>ARE</td>
<td>Algebraic Riccati Equation</td>
</tr>
<tr>
<td>CEV</td>
<td>Centre d’Essais en Vol</td>
</tr>
<tr>
<td>CIRA</td>
<td>Centro Italiano Ricerche Aerospaziali</td>
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<tr>
<td>DAy</td>
<td>Dassault Aviation</td>
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<td>DASA</td>
<td>DaimlerChrysler Aerospace</td>
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<td>DF</td>
<td>Describing Function</td>
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<tr>
<td>DERA</td>
<td>Defence Evaluation and Research Agency</td>
</tr>
<tr>
<td>DLR</td>
<td>Deutsches Zentrum für Luft-und Raumfahrt e.V.</td>
</tr>
<tr>
<td>DTT</td>
<td>Discrete Tracking Task</td>
</tr>
<tr>
<td>DUT</td>
<td>Delft University of Technology</td>
</tr>
<tr>
<td>EG</td>
<td>Exploratory Group</td>
</tr>
<tr>
<td>ES</td>
<td>Spain</td>
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<tr>
<td>FCS</td>
<td>Flight Control System</td>
</tr>
<tr>
<td>FFA</td>
<td>Flygtekniska Försöksanstalten, Sweden</td>
</tr>
<tr>
<td>FM/GoR</td>
<td>Flight Mechanics Group of Responsables</td>
</tr>
<tr>
<td>FOSIM</td>
<td>Forskningsimulator</td>
</tr>
<tr>
<td>FR</td>
<td>France</td>
</tr>
<tr>
<td>GARTEUR</td>
<td>Group for Aeronautical Research and Technology in EURope</td>
</tr>
<tr>
<td>INTA</td>
<td>Instituto Nacional de Tenica Aeroespacial</td>
</tr>
<tr>
<td>IT</td>
<td>Italy</td>
</tr>
<tr>
<td>LAAS</td>
<td>Laboratoire d’Analyse et d’Architecture des Systmes</td>
</tr>
<tr>
<td>LFT</td>
<td>Linear Fractional Transformation</td>
</tr>
<tr>
<td>LMI</td>
<td>Linear Matrix Inequality</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear Time Invariant</td>
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<td>NL</td>
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</tr>
<tr>
<td>NLR</td>
<td>Nationaal Luchtvaart- en Ruimtevaartlaboratorium</td>
</tr>
<tr>
<td>NSF</td>
<td>National Simulation Facility</td>
</tr>
<tr>
<td>OLOP</td>
<td>Open Loop Onset Point</td>
</tr>
<tr>
<td>OLT</td>
<td>Offset Landing Task</td>
</tr>
<tr>
<td>ONERA</td>
<td>Office National d’études et de Recherches Aérospatiales</td>
</tr>
<tr>
<td>PIO</td>
<td>Pilot-Involved (or Pilot-Induced / Pilot-in-the-Loop) Oscillations</td>
</tr>
<tr>
<td>PRS</td>
<td>Position Rate Saturation</td>
</tr>
<tr>
<td>PVS</td>
<td>Pilot-Vehicle System</td>
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<tr>
<td>Abbreviation</td>
<td>Full Form</td>
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<td>ROBAN</td>
<td>Robust Stability Analysis</td>
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<tr>
<td>SAAB</td>
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</tr>
<tr>
<td>SE</td>
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</tr>
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<td>SMA</td>
<td>Saab Military Aircraft</td>
</tr>
<tr>
<td>SUE</td>
<td>Super-Etendard Simulator</td>
</tr>
<tr>
<td>TBC</td>
<td>To Be Confirmed</td>
</tr>
<tr>
<td>TBD</td>
<td>To Be Defined</td>
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<td>UNAP</td>
<td>Università degli Studi di Napoli Federico II</td>
</tr>
<tr>
<td>WP</td>
<td>Work Package</td>
</tr>
<tr>
<td>XC</td>
<td>GARTEUR Executive Committee</td>
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1 Introduction

The GARTEUR action group FM(AG12) ‘Pilot-in-the-Loop Oscillations; Analysis and Test Techniques for their Prevention’ conducts analytical and experimental investigations in the area of nonlinear PIO. The research is oriented towards developing analysis and test procedures which prove that a given highly augmented aircraft is sufficiently free from PIO proneness, with an emphasis on Category II PIOs.

The GARTEUR report [FM/99] has provided a consolidated view on the applicability of the existing most prominent set of Category I and II PIO susceptibility criteria. It can be seen that pass/fail assessment criteria for Category II PIOs are not as well developed as for Category I PIOs. The most well known are the time-domain Neal-Smith criterion and the OLOP criterion. In addition to these major findings, other elaborate analyses need to be found to correlate the results and to propose a mixed approach which can predict with absolute accuracy the presence or absence of Category II PIO potential.

Four analysis approaches are proposed by the GARTEUR action group FM(AG12) in this report:

- Bifurcation analysis
- Analysis using classical and extended mu tools
- IQC analysis
- Robust stability analysis

One of the primary benefits that is sought from applying these methods is to introduce, in a formal manner, the ability to include model uncertainty and nonlinear behavior of the pilot-vehicle system. A more generalized, multivariable interpretation of stability robustness, and the possibility to search for specific pilot behavior and inputs that destabilize the combined pilot-vehicle system may offer useful indicators and criteria to be met in flight control system design.

This report is divided into four parts, each dealing with one particular analysis method. Each part starts with a short tutorial on the background theory, followed by a description of how the general problem of Category II PIO is formulated such that the theory can be applied to provide more insight. In all cases, the theoretical work is supported by application on a series of test cases taken from recent experimental studies into nonlinear PIO: the SAAB data base and the FOSIM data base. These evaluations make it possible to compare the results of the analysis with other criteria and, in some cases, with experimentally obtained results. Also, practical issues like visualization and computational cost of the different methods are covered. The four parts are preceded by a chapter describing the testcases that are used for evaluation of the different methods. The description includes the results of two prominent PIO prediction criteria. The reports ends with some
general conclusions on the insights gained from the work and the potential of the different methods for use as PIO prediction tools.

The work contained in this report is by no means meant as a definite direction in which PIO research and evaluation shall head. It is intended, rather, as a study into the potential of elements of modern control theory in solving the complex problem of PIO prediction and prevention.
2 Description of Handling Qualities databases

The four methodologies that are introduced in this report will be tested and validated on a number of test cases that have previously been evaluated in other studies. Some of these configurations have been test flown using in-flight simulation, others evaluated using ground-based simulation. Generally speaking, these configurations and their Category I and II PIO characteristics are well understood. In this chapter, the test cases are introduced and results of earlier evaluations are presented briefly. For all cases, the traditional Gibson Phase Rate criterion is applied to test for Category I PIO. Also, the results of the application of the Open-Loop Onset Point (OLOOP) criterion, developed at DLR [Dud97a], are provided. The OLOOP criterion was designed specifically to deal with the prediction of Category II PIO and has been validated by using a wide range of cases after its introduction in 1997. The results of the OLOOP criterion can serve as a reliable base for comparison of the different newly introduced analysis methods presented in this report.

The information contained in this chapter is reasonably general; more detailed information on the different cases and flight control systems are provided in the specific sections of this report in which the configurations are applied to test the various analysis methods.

2.1 Definition of the aircraft models from a FOSIM study

In a recent study by the German DLR and Swedish FFA [DD98], flight simulator experiments were conducted to specifically study Category II PIO. A large number of aircraft models from the LATHOS [MSR82], F-18 [Smi79b] and YF-16 [Smi79a] data bases have been evaluated. Some representative configurations were selected from these data bases and extended to rate limiters in the forward path and the feedback loop of their roll axis flight control systems respectively. The maximum rates of the limiters were defined in order to cover a wide spectrum regarding the OLOOP criterion. Configurations with different Category I PIO characteristics were purposely selected to study the relationship between Category I and II PIO. Some modifications on the flight control systems were introduced in order to get a similar roll performance and to compensate for the simulator time delay of 50 ms. Table 2.1 presents the selected configurations.

Two basic configurations from each data base were selected for simulator testing. The file indices L1 to Y2 will be used subsequently. In figure 2.1 the roll rate step responses for the maximum stick force of 80 N are presented.

All aircraft models have a similar roll performance, but the YF-16 models are characterized by an extremely adverse roll behavior. A roll oscillation with poor damping ($\zeta = 0.3$) occurs and the roll rate returns to zero after about ten seconds although the roll stick is still fully deflected. An explanation for this very strange roll behavior might be that the linear YF-16 model used was only valid in the ground effect. It is obvious that no pilot would like this behavior, but nevertheless it was decided to evaluate the YF-16
<table>
<thead>
<tr>
<th>Database</th>
<th>Configuration</th>
<th>Index</th>
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<tr>
<td>LATHOS</td>
<td>L3.2</td>
<td>L1</td>
</tr>
<tr>
<td></td>
<td>L2.1T4</td>
<td>L2</td>
</tr>
<tr>
<td>F-18</td>
<td>1.0</td>
<td>F1</td>
</tr>
<tr>
<td></td>
<td>B1</td>
<td>F2</td>
</tr>
<tr>
<td>YF-16</td>
<td>INI</td>
<td>Y1</td>
</tr>
<tr>
<td></td>
<td>MOD</td>
<td>Y2</td>
</tr>
</tbody>
</table>

*Table 2.1 Selected configurations from FOSIM study*

configurations on FOSIM. In table 2.2 the basic configurations to be evaluated and the maximum rates of the limiters are summarized. Their PIO characteristics are illustrated in figure 5 utilizing the phase rate criterion (Category I) and OLOP criterion (Category II). In both criteria the additional time delay of the simulator (50 ms) is considered. In the OLOP criterion a pilot model gain spectrum is considered (crossover phase angles from $\Phi_{cr} = -120^\circ$ to $\Phi_{cr} = -160^\circ$). This gain spectrum elucidates the pilot gain sensitivity of each configuration. For instance, L1130 is located in the critical area only for high gains and relocates directly into the safe area when the pilot reduces his gain. F1140 on the other hand even for low gains is located in the critical area and is therefore prone to PIO due to rate saturation for all gains.

![80 N Step Responses](image)

*Fig.2.1 Roll stick step responses of the LATHOS, YF-16 and F-18A configurations*
Fig. 2.2 PIO characteristics of the LATHOS, YF-16 and F-18A configurations

<table>
<thead>
<tr>
<th>Index</th>
<th>Maximum Rate</th>
<th>Extended Index</th>
<th>Comments / Expectations</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>130 N/sec</td>
<td>L1130</td>
<td>No Category I PIO, rate limiter in the forward path, clear Category II PIO potential for high pilot gains predicted.</td>
</tr>
<tr>
<td>L1</td>
<td>220 N/sec</td>
<td>L1220</td>
<td>No Category I PIO, rate limiter in the forward path, little Category II PIO potential for high pilot gains predicted.</td>
</tr>
<tr>
<td>L2</td>
<td>110 N/sec</td>
<td>L2110</td>
<td>Strong Category I PIO, rate limiter in the forward path, little Category II PIO potential for high pilot gains predicted.</td>
</tr>
<tr>
<td>F1</td>
<td>140 deg/sec</td>
<td>F1140</td>
<td>No Category I PIO, rate limiter in the feedback loop, strong Category II PIO even for low pilot gains predicted.</td>
</tr>
<tr>
<td>F1</td>
<td>200 deg/sec</td>
<td>F1200</td>
<td>No Category I PIO, rate limiter in the feedback loop, no Category II PIO predicted.</td>
</tr>
<tr>
<td>F2</td>
<td>110 deg/sec</td>
<td>F2110</td>
<td>No Category I PIO, rate limiter in the feedback loop, little Category II PIO potential for high pilot gains predicted.</td>
</tr>
<tr>
<td>Y1</td>
<td>75 deg/sec</td>
<td>Y1075</td>
<td>Medium Category I PIO, but adverse roll behavior, rate limiter in the feedback loop, strong Category II PIO even for low pilot gains predicted.</td>
</tr>
<tr>
<td>Y1</td>
<td>90 deg/sec</td>
<td>Y1000</td>
<td>Medium Category I PIO, but adverse roll behavior, rate limiter in the feedback loop, Category II PIO potential predicted.</td>
</tr>
<tr>
<td>Y2</td>
<td>90 deg/sec</td>
<td>Y2000</td>
<td>Medium Category I PIO, but adverse roll behavior, rate limiter in the feedback loop, Category II PIO potential for high pilot gains predicted.</td>
</tr>
</tbody>
</table>

Table 2.2 Configuration summary (from: [DD98])
An important choice that was made for the F-18A configurations used in the FOSIM study involves the incorporation of position and rate limits in the control system. These nonlinear elements are included as software limiters, i.e. they do not serve as elements that represent the actual physical limitations of the actuators but as protecting elements resident in the flight control laws. The order in which these elements are placed are of concern. On real aircraft, the position limit should always be placed in front of the rate limit, as indicated in Figure 2.3a. Phase loss during simulations rate and position limiting is exacerbated when this order is swapped, as shown in Figure 2.3b. In the FOSIM study, however, this latter order was applied for experimental purposes.

![Diagram](attachment:image.png)

a) Required order to prevent adverse effects  
b) Order applied in FOSIM for experimental purposes

**Fig. 2.3 Order of nonlinearities**

This leads to a choice to be made by the analysis teams that present their results in this report. The analysis can be either applied on the ‘correct’ order, such that the results will represent realistic nonlinear effects that are experienced in real-flight. Alternatively, the ‘experimental’ architecture used in FOSIM can be used, such that the results of the analysis can be correlated with the observations and ratings obtained in the FOSIM study. The analysis teams were held free to choose which architecture to use, and this choice is presented each time the F-18A configuration is presented.
2.2 Definition of the aircraft model from SMA study

An additional test configuration is taken from a study performed by SMA Saab Military Aircraft [Run95]. During in-flight simulation using the Calspan Learjet the behavior of different types of phase-compensation rate limiters was evaluated. In the cases presented here, a conventional rate limiter is positioned in the forward path of the flight control laws (right after the stick command). The value of the rate limit was chosen to be 10 deg/s, which is relatively low but necessary to introduce critical Category II PIO tendencies.

In the flight experiments, both Discrete Tracking Tasks (DTT) and Offset Landing Tasks (OLT) were flown. Although different configurations were flown during the program, we will confine ourselves to the one case of 10 deg/s of conventional rate limiting. This configuration was evaluated by different pilots while flying the same DTT. Table 2.3 lists four runs performed with three different pilots (537, 538 and 539) together with the ratings and some comments of observations made in-flight.

<table>
<thead>
<tr>
<th>Pilot/Run</th>
<th>Task</th>
<th>CHR/PIOR</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>537/04</td>
<td>DTT</td>
<td>8/4</td>
<td>PIO lasts over 3 periods. Roll stick command signal completely out of phase with roll angle.</td>
</tr>
<tr>
<td>538/02</td>
<td>DTT</td>
<td>7/4</td>
<td>High roll stick activity, roll stick command completely out of phase before onset.</td>
</tr>
<tr>
<td>539/02</td>
<td>DTT</td>
<td>7/4-5</td>
<td>Over control, followed by divergent PIO, not divergent after adaptation from pilot</td>
</tr>
<tr>
<td>539/08</td>
<td>DTT</td>
<td>5/3</td>
<td>The pilot has adapted himself to the aircraft</td>
</tr>
</tbody>
</table>

*Table 2.3 SMA evaluation data (from: [Dud97a])*

For analysis purposes, linear models of the combined aircraft and flight control system dynamics were developed in [Dud97a]. Based on the data obtained in-flight with rate limiters switched off, frequency domain identification techniques were used yielding frequency response descriptions of the linear aircraft dynamics. Both these data, and the actual experimental data obtained in-flight were made available for analysis for the purpose of this GARTEUR study. Step responses to full roll stick deflections of the linear models are presented in figure 2.4. Although the exact same configuration was flown, the identification process yields a varying set of time responses. This is attributable to differences in flight conditions between the four runs, as well as the inherent uncertainty involved with frequency response estimation from flight data. So realistically, the actual aircraft response is likely to be 'somewhere' in the range contained by the different plotted responses. Phase rate and OLOP criteria mappings are shown in figure 2.5. For the same reason as above, the mappings differ for the different runs. Again, this should be interpreted as a range within which the real aircraft is likely to fall.
Fig. 2.4 Roll stick step responses of the SMA configurations

Fig. 2.5 PIO characteristics of the SMA configurations
PART I

Bifurcation Analysis
3 Tutorial on method

3.1 Analysis of continuous nonlinear dynamical systems

3.1.1 Equilibrium and stability of continuous systems

The continuous nonlinear dynamical systems are represented as ordinary nonlinear differential equations:

\[ \dot{x} = f(x, \lambda) \]  

(3.1)

where

- \( x \) is the \( n \)-dimensional state vector
- \( \lambda \) is the \( m \)-dimensional parameter vector
- \( f \) is a vector of \( n \) nonlinear continuous and differentiable functions

As opposed to linear systems, a nonlinear differential system may have several asymptotic states for a given set of parameters. In the simplest cases, as generally in flight dynamics, the asymptotic states are fixed points corresponding to the nonlinear equation:

\[ 0 = f(x, \lambda) \]  

(3.2)

In other cases, the asymptotic states are periodic orbits called limit cycles, satisfying:

\[ 0 = \int_0^T f(x, \lambda)dt \]  

(3.3)

with a period \( T \) which is an additional unknown. It has to be noticed that the parameters are here considered constant in time. The results obtained when the temporal variations of the parameters are not negligible can be different from those predicted in the frame of bifurcation theory.

The study of the stability of the fixed points and of the periodic orbits is the main goal of this theory. Its mathematical background can be found in many specialized books [IJ81],[MMC76], [GH83]. The fixed points are said stable if all the eigenvalues of the linearized system have a negative real part. The stability of the periodic orbits is established by the Floquet theory, considering the eigenvalues of the transition matrix: if all the eigenvalues (except one which is equal to 1) have a module less than unity, the orbit is stable. The different kinds of bifurcations from fixed points and from periodic orbits will be summarized hereafter. The following paragraphs are partly extracted from the Agard LS 191 [Gui93], in which the essential mathematical background of the theory is also presented.

3.1.2 Bifurcation of fixed points

The most frequent phenomenon that occurs with fixed points corresponds to the case where a real eigenvalue of the linearized system changes sign (from negative to positive)
as a parameter is varied. Two cases can happen (Fig. 3.1):

- If the fixed point is regular, i.e. the implicit function theorem works, the equilibrium curve exhibits a limit point, or regular turning point. The solution curve is unique.

- If the fixed point is irregular, i.e. the implicit function theorem does not work, new equilibrium branches with distinct tangents can appear. Various bifurcation cases are then possible, depending on the dimension and symmetry of the system.

Another frequent case of bifurcation from fixed-points is the Hopf bifurcation: it happens when two conjugate imaginary eigenvalues cross the imaginary axis as one of the parameters varies while the other eigenvalues remain in the left half plane. This situation corresponds to the apparition of closed orbit in addition to the fixed point. Depending on the system's characteristics, this bifurcation may be supercritical or subcritical (Fig. 3.2), leading to different behaviors in the vicinity of the bifurcation point. Crossing a supercritical Hopf bifurcation leads to a smooth divergence to a limit cycle with a small amplitude. Coming back to the initial value of the parameter, the fixed point is reached again in a continuous way. When the bifurcation is subcritical the system can exhibit a sudden
Fig. 3.2 Hopf bifurcation

3.1.3 Stability of periodic solutions

The stability of a closed orbit can be characterized by the Floquet theory. A linearization of the dynamical system about the closed orbit leads to define a transition matrix defined as:

\[ H = \left[ \frac{\partial \int_0^T f(x, \lambda) dt}{\partial x} \right] \]  \hspace{1cm} (3.4)

where \( T \) is the period of the orbit. The eigenvalues of this transition matrix are called the Floquet multipliers. The multiplier associated with perturbations along the closed orbit is always equal to 1. The remaining eigenvalues determine the stability of the orbit. From a practical point of view, \( H \) can be obtained by generating a set of \( n \) linearly independent
perturbed solutions of $f(x)$.

3.1.4 Bifurcation from periodic orbit

A bifurcation case which is of great interest in flight dynamics is when the stability of a periodic orbit is changed as a parameter is varied. This phenomenon is always associated to the fact that one or two multipliers move outside the unit circle (Fig. 3.3). Three main causes can be found:

- One real multiplier crosses through the unit circle at +1. In that case a turning orbit appears. The situation is analogous to bifurcation from fixed point with eigenvalue 0.

- One real multiplier crosses through the unit circle at -1: this bifurcation is referred as “period doubling bifurcation”, “flip bifurcation”, “Brunovsky bifurcation” or “subharmonic bifurcation”. At this point, the originally stable periodic orbit becomes unstable and a branch of periodic solutions with a two fold period branches off. Several types of period doubling bifurcations exist (supercritical or subcritical). They also concern unstable orbits. Moreover, the period doubling bifurcation often occurs repeatedly (Feigenbaum cascade) leading to complex orbit associated to very long period and then to chaotic (non periodic) motion.

- Two conjugate eigenvalues intersect the unit circle. At this point, the originally stable orbit becomes unstable and a stable or unstable torus may appear.

Other complex cases exist, where several multipliers cross the unit circle simultaneously, but they are not described here.

3.2 Numerical procedures

The computation of the fixed points and limit cycle characteristics is performed with a continuation algorithm. Continuation methods are a direct result of the implicit function theorem which states that if the Jacobian of a non-linear system at a known fixed point is non-singular, then there exists a unique curve of fixed points containing the known fixed point.

Starting from a given value of the parameter, the continuation algorithm is used to track the zeros of the system. The case of singular Jacobian is treated by augmenting the state vector with the parameter, thus allowing to solve the system of $n$ linear algebraic equations with $n + 1$ unknowns by means of Gauss elimination [GH83].

A particular attention has to be put on the treatment of local discontinuities of the dynamics, as for example for the modeling of the saturations or other nonlinearities. If necessary, the original functions may be smoothed in order to achieve good continuity and derivability properties.
Fig. 3.3 Sketch of the main bifurcations from periodic orbits
Fig.3.4 Augmented system for PIO analysis through bifurcation methodology

Some relevant available packages based on continuation methods are: ALCON [DFK87], ASDOBI [Gui92], AUTO [DK86], BIFPACK [Sey89], BISTAB [WK84], CONEX [ROSM90], CONKUB [Mei86], CONSOL [Mor87], HOMPACK [WBM87], KRIT [GK93], LINLBF [Khi90], PATH [KP87], PITCON [RB83], PLTMG [Ban88].

3.3 Application of bifurcation to PIO

The application of bifurcation methods to the analysis of pilot induced oscillations was proposed in 1997 in the framework of one of the follow-on research activities of GARTEUR FM(AG08) [DV97]. The first results on PIO analysis using this technique are however published in 1998 by Mehra and Prasanth [RR98]. The main features of their approach are described hereafter.

The first step of the analysis is the problem formulation: the whole pilot-vehicle system (PVS) has to be formulated in a state-space form:

$$\dot{x} = f(x, u_c)$$  \hspace{1cm} (3.5)

In this formulation, the PVS is a forced nonlinear system which cannot be studied with a standard bifurcation method: the problem is treated by augmenting the PVS with a nonlinear oscillator which has the periodic command as an asymptotically stable solution (Fig. 3.4). As there is no feedback from the PVS to the oscillator, it can be shown that the global properties of the PVS are the same as those of the augmented system.

The oscillator proposed by Mehra and Prasanth is given by:

$$\dot{x}_1 = x_1 + \omega x_2 - x_1(x_1^2 + x_2^2)$$  \hspace{1cm} (3.6)

$$\dot{x}_2 = -\omega x_1 + x_2 - x_2(x_1^2 + x_2^2)$$  \hspace{1cm} (3.7)
where $\omega$ is the command input frequency. It can be shown that the solution is asymptotically stable. The augmented system (NLO + PVS) can be written as:

\[
\begin{bmatrix}
\dot{x}_{NLO} \\
\dot{x}_{PVS}
\end{bmatrix}
= 
\begin{bmatrix}
f_1(x_{NLO}; \omega, A) \\
f_2(x_{NLO}, x_{PVS}; u_{PVS})
\end{bmatrix}
\]  
(3.8)

where $x_{NLO}$ and $x_{PVS}$ are the state vectors for nonlinear oscillator and PVS, $\omega$ and $A$ are command input frequency and amplitude, and $u_{PVS}$ is a vector of parameters for the PVS (pilot gain, delays, saturation values, ...). The above augmented system can then be studied by means of a bifurcation approach in the second step. For the analysis of the PIO features, the limit cycles have to be computed, one of the parameters being chosen as continuation parameter, the others being fixed.

This approach was successfully applied by Mehra and Prasanth to two kind of nonlinear PVS:

- a linear airframe with a rate limited actuator,
- and an airframe with nonlinear aerodynamics.

The limit cycle amplitude was computed as a function of pilot gain, showing a large jump corresponding to the onset of nonlinear effects and PIO.
4 Description of analysis set-up

For the GARTEUR Analysis Challenge, an approach similar to Mehra and Prasanth’s [RR98] is used to assess PIO potential, with the following steps:

- Formulate a limit cycle and bifurcation problem for the PVS. Compute bifurcation surfaces and limit cycles as a function of pilot gain and other parameters of interest.

- Check for Hopf bifurcations and jump resonances leading to limit cycles and large jumps in limit cycle amplitude as the pilot gain increases. Flying qualities “cliffs” are associated with these nonlinear phenomena.

In what follows, a description of the PVS is given, the nonlinear elements and the relevant parameters are recalled.

4.1 Aircraft and pilot models

The aircraft model used is a lateral model of the F-18 obtained from the F-18A in-flight simulation program which was conducted using the variable stability NT-33 aircraft, [Smi79b]. The Simulink model of the configuration from the database is presented in figure 4.1. Note that the rate and position limits indicated in the figure were not originally implemented on the NT-33. They have been added for the purpose of a simulation study conducted by DLR and FFA that specifically dealt with Category II PIO, see [DD98]. It should also be noted that the order in which the saturation elements are positioned is not optimal. Normally, these elements are implemented such that a position limiter is encountered before a rate limiter. This way, adverse rate limiting effects are alleviated.

During a sustained PIO, the pilot’s behavior and response is intricately coupled with the dynamics of the airplane. A model of the human pilot is therefore essential. There are several possible analytical approaches to pilot modeling, the simplest being the classical crossover model. For this application, a pure gain will be used to represent the human pilot. This assumption is well accepted in the case where the PIO is fully developed (Synchronous Precognitive Behavior [McR95]).

The closed-loop model of the augmented aircraft-pilot system is presented on figure 4.2. Note that a limitation on the stick force has been added.

The PVS is augmented with a nonlinear oscillator which has the periodic command as an asymptotically stable periodic solution. The augmented system is an unforced dynamical system and can be analyzed using standard bifurcation methodology.

From the Simulink model, a state-space model that represents the bare aircraft, the flight control system, the pilot and the nonlinear oscillator is determined. This yields a 18th-order nonlinear system.
4.2 Parameters of interest

Only the configuration F1 of the F18 database is considered within this study. The relevant parameters are:

- the aileron rate limit
- the pilot gain
- the command frequency
- the command amplitude

Three values for rate limits are used as in [DD97]: 200 deg/sec, 140 deg/sec and 80 deg/sec. The maximum and minimum deflection limits of the aileron are +40 deg and -40 deg. The maximum and minimum force limits of the stick are +80 N and -80 N.

During a sustained PIO event, the pilot will increase his gain to a level that is significantly higher than normal. If the event occurs near the ground or near other airplanes, one would expect the pilot to apply abrupt control in an attempt to quickly regain a steady flight condition. It is difficult to determine the appropriate pilot gain factor in such an agitated state. The pilot model gain are assumed to be adjusted based on the linear crossover phase angle of the open loop aircraft-pilot system $\Phi_c$. Within this study a gain spectrum from $\Phi_c = -110$ deg (low pilot gain) up to $\Phi_c = -160$ deg (high pilot gain) is used (see Table 4.1).

The command frequency is varied in the range from 1 rad/sec to 3 rad/sec, while the amplitude is varied between 10 deg and 30 deg.
Fig. 4.2 Aircraft-pilot loop

<table>
<thead>
<tr>
<th>Φc (deg)</th>
<th>-110</th>
<th>-120</th>
<th>-130</th>
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<td>1.83</td>
<td>2.60</td>
<td>3.54</td>
<td>4.70</td>
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</table>

Table 4.1 Pilot gains

Each case will be denoted Cxxxyzz where xxx is the rate limit in deg/sec, y the command frequency in rad/sec and zz is the command amplitude in deg.
5 Results of analysis

The state trajectory of a stable limit cycle will return to the limit cycle after it has been perturbed away from it. The state trajectory of an unstable limit cycle will not return to its solution after perturbation. Most PIOs are sustained for several cycles, so stable limit cycles in the pilot/vehicle system are sought. The limit cycle is predicted when one pair of eigenvalues is purely imaginary. Figure 6.1 to Figure 6.24 present the computed envelopes and amplitudes of the stable limit cycles, as a function of the pilot gain. In most cases, the limit cycles can be found without numerical problems by the ASDOBI software. For some cases, the asymptotic solution is more complex (period doubling bifurcation,...) and a failure of the limit cycle search by ASDOBI is encountered. For those cases, a simulation is performed and the “envelope” of the trajectory is plotted.

For small pilot gains $K_{pl}$, the amplitude is that of a periodic solution, depending on the command frequency and amplitude. As $K_p$ increases, a Hopf bifurcation is encountered leading to large jumps in amplitude and limit cycles. The parameters combinations for which a large jump in limit cycle amplitude is observed are summarized in table 5.1. A cross in the table indicates that a jump in limit cycle amplitude has occurred as the pilot gain is increased. For rate limit 200 deg/sec, this can be already observed for pilot gain $K_{pl} = 2.6$ or crossover phase angle $\Phi_c = -120$deg. For 140 deg/sec and 80 deg/sec, this happens for lower pilot gain $K_{pl} = 1.82$ or $\Phi_c = -110$deg. In principle, a jump indicates a significant change in PVS structural stability and should correspond to the onset of PIO. It is not possible at this preliminary stage to make a correlation with the OLOP criterion, because further analysis of the results is necessary. It is only reminded that for Configurations F1080 and F1140 [DD97], the OLOP criterion predicts strong Category II PIO even for low pilot gains, and for Configuration F1200, no Category II PIO is predicted even for high gains.
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Table 5.1 Effect of pilot gain on limit cycle stability
6 Conclusions, Part I

An approach for the analysis of nonlinear PIO using bifurcation techniques has been presented. The approach involves computation of nonlinear phenomena such as Hopf bifurcations that result in large changes in structural stability. An augmentation technique to convert the PVS which is a forced system to a system amenable to bifurcation analysis is presented. This is done using an asymptotically stable nonlinear oscillator in such a way that the global stability properties of the forced PVS are unchanged. The limit cycle amplitudes are computed as a function of pilot gain. Preliminary results obtained with a lateral model of the F-18 aircraft show a large jump in limit cycle amplitude as the pilot gain is increased, indicating a significant change in PVS structural stability. The results also give the combinations of rate limit, command frequency and amplitude leading to this stability change. Further analysis work will be necessary to evaluate the potential of the approach.
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 10 deg

Fig.6.1 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 20 deg

Fig.6.2 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 30 deg

Fig. 6.3 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 10 deg

Fig.6.4 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 20 deg

Fig. 6.5 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 30 deg

Fig.6.6 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 10 deg

Fig.6.7 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 20 deg

Fig.6.8 Limit cycle diagram
RATE LIMIT 200 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 30 deg

Fig.6.9 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 10 deg

Fig.6.10 Limit cycle diagram
Fig. 6.11 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 30 deg

Fig.6.12 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 10 deg

Fig.6.13 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 20 deg

Fig. 6.14 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 30 deg

Fig.6.15 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 10 deg

Fig. 6.16 Limit cycle diagram
Fig. 6.17 Limit cycle diagram
RATE LIMIT 140 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 30 deg

Fig.6.18 Limit cycle diagram
RATE LIMIT 80 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 10 deg

Fig. 6.19 Limit cycle diagram
RATE LIMIT 80 deg/sec COMMAND FREQUENCY 1 rad/sec AMPLITUDE 20 deg

Fig. 6.20 Limit cycle diagram
Rate limit 80 deg/sec command frequency 2 rad/sec amplitude 10 deg

Fig. 6.21 Limit cycle diagram
RATE LIMIT 80 deg/sec COMMAND FREQUENCY 2 rad/sec AMPLITUDE 20 deg

Fig. 6.22 Limit cycle diagram
RATE LIMIT 80 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 10 deg

Fig. 6.23 Limit cycle diagram
RATE LIMIT 80 deg/sec COMMAND FREQUENCY 3 rad/sec AMPLITUDE 20 deg

Fig. 6.24 Limit cycle diagram
PART II

Analysis using classical and extended mu tools
Summary

Combination of position and rate saturations in modern flight control systems is known to be the main contributor to the triggering of Pilot-In-the-loop-Oscillations (PIO). The analysis technique which we propose here to predict wether or not a pilot-vehicle system is PIO prone then relies on the study of saturation effects. As in “describing function analysis”, the non-linearities are replaced by their equivalent gains. An LFT model of the system is then derived which permits the use of μ tools. These are slightly adapted here in order to detect potential limit-cycles.

The proposed analysis method is applied on FOSIM nonlinear database and gives similar results as those obtained via the OLOP criterion. As far as numerical aspects are concerned, the method performs well. Moreover, extensions to multivariable systems can be considered.
7 Introduction

The introduction of digital fly-by-wire flight control systems has increased the potential for adverse interactions between the human pilot and the aircraft dynamics. These interactions have long been called Pilot-Induced-Oscillations (PIO). But it was recently established that the observed oscillations result from coupling effects between the pilot and the system. Consequently, the phenomenon was renamed A.P.C. (Aircraft-Pilot-Coupling). Yet, everybody is used to PIO initials and finally this name was kept, but PIO now means: Pilot-In-the-loop-Oscillations.

The review of historical incidents and accidents in [KMM95] revealed that rate limiting is the dominating non-linear effect in modern flight control systems leading to severe flying qualities cliffs. It was observed that in all PIO catastrophic incidents, the effects of actuator rate limiting were present.

To facilitate the understanding of PIO through a systematic study, a classification was recently introduced:

- **Cat. I** Essentially linear pilot-vehicle oscillations,
- **Cat. II** Quasi-linear pilot-vehicle oscillations with rate or position limiting,
- **Cat. III** Essentially non-linear pilot-vehicle oscillations, such as multiple non-linearities, transitions in pilot behavior, etc.

The proposed analysis methodology which is reported here, permits to study **Cat II** PIO and partly **Cat III**, since multiple non-linearities can be handled. To be more precise, we will mainly focus on an “extended Cat II” case, considering the effects of simultaneous position and rate limiting.
8 Description of the model used

8.1 A short description of the open-loop

The model used in this report is extracted from FOSIM database which contains data and models form the experiments conducted by DLR and FFA for the validation of the OLOP criterion. For further informations on these experiments and the models can be found in [Dud97b, Dud97c].

The model used in our study is a lateral model of F-18 aircraft. It was obtained from the F-18A in-flight simulation program [Smi79b] which was conducted using the variable stability NT-33 aircraft prior to first flight of the F18-A. The Simulink model is shown on figure (8.1).

![Diagram of F-18 model with FCS](image)

Fig.8.1 F-18 model with FCS; Order of position and rate limit elements is switched intentionally in order to correlate with the FOSIM study

8.2 The pilot model

As is done with the OLOP analysis method, the pilot behavior will be modeled by a simple gain. It has been remarked indeed that, during fully developed PIO, the pilot usually reacts as a simple gain. This gain is adjusted based on the linear crossover phase angle of the open-loop aircraft-pilot system. Within this study, a gain spectrum from $\phi_c = -110^\circ$ (low pilot gain) up to $\phi_c = -160^\circ$ (high pilot gain) is used.

The closed-loop Simulink model of the aircraft-FCS-pilot system is shown on figure (8.2). The different pilot gains that will be used in the analysis are given in table 8.2, which also displays the associated crossover phase angles.
\[ \begin{array}{|c|c|c|c|c|c|} \hline \phi_c \ (\text{deg}) & -110 & -120 & -130 & -140 & -150 & -157.5 \\ \hline K_{pil} & 1.18 & 1.83 & 2.60 & 3.54 & 4.7 & 5.74 \\ \hline \end{array} \]

Table 8.1 Pilot gains

Fig.8.2 F-18 model + Pilot - Closed-loop system; Order of position and rate limit elements is switched intentionally in order to correlate with the FOSIM study

8.3 Why this model?

The reason why this model was chosen is that it simultaneously exhibit rate and position limiting. Therefore, such a model was supposed to fall well in the scope of the analysis technique that we will now describe.
9 Description of the analysis technique

The core of the proposed analysis technique relies on $\mu$ bounds computations. Consequently, the non-linear system to be analyzed, which here only includes saturation-type non-linearities must be first rewritten as an LFT. A solution to this preliminary issue is first proposed. Then, we recall the principle and the main result of multivariable describing function analysis, that is the necessary condition of oscillation. Finally, links with $\mu$ analysis are highlighted and the proposed analysis method is described.

9.1 An LFT model for saturated systems

9.1.1 A short background

As is well-known, a position limiter can be easily replaced by an equivalent time-varying gain $\delta(t)$ evolving in the interval $[0, 1]$. To be more precise, as shown on figure (9.1), the equivalent gain, depends on the input signal $u$. Therefore, the time-dependence is implicit:

$$\delta(t) = \delta(u(t)) = 1 - \min \left( 1, \frac{L_p}{|u(t)|} \right)$$

![Diagram](image)

*Fig.9.1 LFT model of a position limiter*

Note that the obtention of the LFT from figure (9.1-b) is straightforward. The approach - which now can be qualified as fairly standard - consists of writing the transfer between fictitious input ($w_\delta$) and output ($z_\delta$) associated to the "perturbation block" $\delta$. Remark that the following relationship between $w_\delta$ and $z_\delta$ must be verified:

$$w_\delta = \delta(u).z_\delta$$

LFT models are widely used in robustness analysis since they provide a unified way of representing different type of uncertainties (time-varying, time-invariant, complex non-structured, parametric-real). We have just shown above that LFT models could also be used to represent saturation-type non-linearities. Yet, it should be underlined that the non-linear element is still present through the fundamental relationship (9.2). As a
result, the application of any standard robustness analysis technique based on the following simplification of (9.2):

\[ w_\delta = \delta(t) . z_\delta , \quad \delta(t) \in [0, 1] \]  \hspace{1cm} (9.3)

would introduce a large amount of conservatism.

9.1.2 Why using equivalent complex gains?

Our method, which is based on non-standard application of the \( \mu \) tools, is essentially sequential. As a result, the above LFT which involves time-varying “uncertainties” is not applicable. The solution then consists of replacing the non-linear element by its equivalent complex gain, prior to LFT model determination. This approach is summarized by the figure (9.2).

![Fig.9.2 LFT model of a position limiter using complex gain](image)

We recall that the equivalent complex gain determination is based on the following assumptions:

- all signal in the system are sinusoidal and oscillating at the same frequency,
- the linear part of the system behaves as a low-pass filter so that high harmonic components are negligible.

In the case of a position limiter the equivalent complex gain is real, and belongs to the interval \([0, 1]\). If we denote \( L \) the maximum amplitude of the output signal, the gain is given by:

\[ N_L(u_0) = \frac{2}{\pi} \left( \arcsin(y) + y \sqrt{1 - y^2} \right) , \quad y = \min \left( 1, \frac{L}{u_0} \right) \]  \hspace{1cm} (9.4)

Remark: The combination of describing function technique and LFT model is of high interest since the “perturbation” \( \delta(u_0) \) (see figure (9.2-d)) can now be considered as time-invariant by the analysis technique. As a matter of fact, the aim is to detect potential limit-cycles, which are sustained oscillations of constant magnitude. So, when a limit-cycle occurs, \( u_0 \) is constant.
9.1.3 Extension to position and rate limiting

The extension of the above results to the case of position and rate limiter is easy. The central idea consists of first replacing the rate limiter by a position limiter. This can be achieved by introducing a first-order filter, which will make appear the derivative of the rate-limited signal. The approach is illustrated on figure (9.3).

![Diagram](attachment:figure_9_3.png)

*Fig.9.3 Position and rate limiting*

**Remark:** Both sub-figures (9.3-a) and (9.3-b) are equivalent if $\tau \to 0$. From a practical point of view, it suffices to tune the time constant until no significant interactions are observed with the system $G(s)$ when closing the loop.

Once the rate limiter has been replaced as shown above, it becomes very simple to derive an LFT model. The procedure is summarized by figure (9.4). Note that it could be extended to the case of multiple non-linearities without any problem.

9.2 Multivariable describing function analysis

9.2.1 Principle

Describing function approach has close ties with linear frequency-domain analysis. As a matter of fact, this analysis is based upon quasi-linear approximations of the non-linear elements. These approximations, which we have already introduced for saturation-type non-linearities are known as describing functions.

The LFT model of figure (9.4-d) is well-suited for multivariable describing function analysis. Assuming that in this model, each feedback loop is oscillating at the same frequency, the system is marginally stable if there exists a non-zero vector $v(j\omega)$ such that:

$$(I - L(j\omega)\Delta(x))v(j\omega) = 0$$

where $x$ is a vector denoting the amplitude of the input signals. We will further see that the components of $x$ may not be independent, and that the frequency may even be involved. Therefore, even in the case of multiple static non-linearities it is preferable to rewrite equation (9.5) as:

$$(I - L(j\omega)\Delta(x(\omega)))v(j\omega) = 0$$
9.2.2 A necessary condition of oscillation

Equation (9.6) represents the harmonic-balance equations for the system. Solutions to (9.6) predict potential existence of limit cycles. Since \( v(j\omega) = 0 \) is a trivial solution to (9.6), one prefers to replace it by:

\[
det(I - L(j\omega)\Delta(x(\omega))) = 0
\]

which is usually referred to as a **necessary condition of oscillation**.

Several methods have been proposed to solve equation 9.7 [GT79, GN81, CPHW87, PN88]. But it appears that every proposed solution requires a fairly sophisticated search. This is the main reason why we propose here an approach based on structured singular value analysis. It was already shown indeed that these two methods (describing function techniques on the one hand and \( \mu \) techniques on the other hand) are very complementary for the study of PIO. An important advantage of the second approach is that computationally efficient approximations to the structured singular value are well-known.

In the next two paragraphs we establish links between both approaches and underline the practical interest of \( \mu \) lower and upper bounds for limit-cycle analysis.
9.3 Variations on the \( \mu \) lower-bound

9.3.1 A short background on \( \mu \) analysis

Let us consider the standard LFT \( M(s) - \Delta \), where \( M(s) \) is a stable LTI plant and \( \Delta \) is a structured linear time-invariant uncertainty. The aim of \( \mu \) analysis is to determine the maximum size of the uncertainty, beyond which instability occurs. At pulsation \( \omega \), the size is defined as:

\[
k_{\Delta} (M(j\omega)) = \frac{1}{\mu_{\Delta}(M(j\omega))} = \min (k \mid \exists \Delta \in kB\Delta \ with \ det(I - \Delta M(j\omega)) = 0)
\]  

(9.8)

where \( B\Delta \) denotes the unit-ball defined as:

\[
B\Delta = \{ \Delta \mid \|\Delta\| < 1\}
\]  

(9.9)

Assume that \( \mu(M(j\omega)) \) has been calculated for each frequency, then the robustness margin \( k_{\text{max}} \) can be obtained as:

\[
k_{\text{max}} = \min_{\omega} \frac{1}{\mu(M_{\Delta}(j\omega))} = \max_{\omega} \frac{1}{\mu(M_{\Delta}(j\omega))}
\]  

(9.10)

The problem is then to know how to compute \( \mu(M(j\omega)) \). It appears that, in the general case, this is an NP-hard problem. This means that an exact value of \( \mu \) cannot be obtained by a polynomial-time algorithm. To bypass this difficulty, one usually computes upper and lower bounds, for which efficient algorithms exist. Then, the quality of the approximation can be measured by checking the gap between the two bounds.

9.3.2 Why and how computing a \( \mu \) lower-bound in our context?

We recall that computing a \( \mu \) upper-bound consists of finding the largest non destabilizing perturbation, whereas computing a \( \mu \) upper-bound consists of computing the smallest destabilizing perturbation. In robustness analysis, the \( \mu \) lower-bound is often computed only for validation of the upper-bound. But in our context, the interest of the lower-bound is clear. As a matter of fact, determining the (smallest size) destabilizing perturbation \( \Delta^{\ast} \), means that:

\[
det(I - \Delta^{\ast} M(j\omega)) = 0
\]  

(9.11)

or, equivalently:

\[
det(I - M(j\omega)\Delta^{\ast}) = 0
\]  

(9.12)

This last equation is rather similar to the necessary condition of oscillation (9.7). As a result, a \( \mu \)-lower bound determination can be used to solve the equation (9.7) and predict limit-cycle, provided that:

\[
\Delta^{\ast} = \Delta(x(\omega))
\]  

(9.13)
The use of standard lower-bounds computation technique does not permit to enforce equation (9.13) though it could be pointed out that a limit-cycle, if any, is most likely to occur for a “small” $\Delta(x(\omega))$. Several methods for computing a $\mu$ lower-bound are available. The method that we will use here was developed at ONERA. It was chosen in this study for the following two reasons:

- this method does not require a frequency gridding and is therefore computationally cheap,
- it can be easily adapted to account for the constraint (9.13).

The principle of the method which we now shortly describe, is quite simple. It is based on a “pole migration” technique towards the imaginary axis. We observe indeed that the matrix $I - M(j\omega)\Delta$ is singular if and only if the matrix $A(\Delta)$ defined by:

$$
\mathcal{F}_\omega(M(s), \Delta) = \begin{pmatrix}
A(\Delta) & B(\Delta) \\
C(\Delta) & D(\Delta)
\end{pmatrix}
$$

exhibits a pair of self-conjugate eigenvalues on the imaginary axis at pulsation $\omega$. Since the nominal system is supposed to be stable, the eigenvalues of $A(0)$ all belong to the left half-plane. The idea then consists, for each eigenvalue $\lambda_i(\Delta)$, to determine the smallest perturbation $\Delta_i^*$ for which there exists $\omega_i^*$ such that:

$$
\lambda_i(\Delta_i^*) = \pm j\omega_i^*
$$

The determination of $\Delta_i^*$ is performed iteratively by a gradient-type constrained-optimization algorithm, using classical results on eigenvalue sensitivity. To accelerate the resolution of each step, the frobenius norm of $\Delta$ is used, instead of the maximum singular value. As a result, when the imaginary axis is reached, the associated perturbation $\Delta_i^*$ is not guaranteed to be the smallest one. If a reliable estimate of the lower-bound is to be found, further optimization along the imaginary axis can be performed. This step is implemented in the original version of the algorithm, but has been removed in the version we will here, since our main interest is to solve equation (9.7), no matter the size of the perturbation.

The last step of the original algorithm consists of sorting the obtained perturbations $\Delta_i^*$ so as to keep the smallest one. In our context, we recall that perturbations are searched with the following structure:

$$
\Delta = \begin{pmatrix}
\delta_p & 0 \\
0 & \delta_r
\end{pmatrix}
$$

and every solution $\Delta_i^*$ which is a priori admissible\footnote{We have shown that $\delta_p$ and $\delta_r$ both necessarily belong to $[0, 1]$. Therefore, the diagonal perturbation $\Delta_i^*$ is said to be a priori admissible if its components $\Delta_i^*(1, 1)$ and $\Delta_i^*(2, 2)$ also belong to $[0, 1]$} will be kept in the outputs of the modified $\mu$-lower bound algorithm.
Now, we will focus more closely on the relation between $\delta_p$, $\delta_r$ and $\omega$, in the particular context of simultaneous rate and position limitations. To take a closer look on the particular relative positions of rate and position limiters, let us redraw hereafter (see figure (9.5)) part of the figure (9.3-b).

Fig.9.5 Relative positions of the limiters

Let us assume that the rate saturation degree $\delta_r = 1 - N(x_r)$ is fixed. Since both limiters are only separated by an integrator, we observe that the amplitude $x_p$ at pulsation $\omega$ is given by:

$$x_p(\omega) = \frac{N(x_r)x_r}{\omega} = \frac{f(\delta_r, L_r)}{\omega}$$  \hspace{1cm} (9.17)

Note that $f(\delta_r, L_r)$ is uniquely defined by:

$$f(\delta_r, L_r) = (1 - \delta_r)N_{L_r}^{-1}(1 - \delta_r)$$  \hspace{1cm} (9.18)

where $N_{L_r}^{-1}(.)$ denotes the reciprocal function of $N_{L_r}(.)$ defined by equation (9.4).

From equation (9.17), we then obtain:

$$\delta_p = 1 - N_{L_p}(x_p(\omega)) = 1 - N_{L_p}\left(\frac{f(\delta_r, L_r)}{\omega}\right)$$  \hspace{1cm} (9.19)

which clearly proves that $\delta_p$, $\delta_r$ and $\omega$ are linked. We note that in the case of significant rate saturation degree, the equation (9.19) can be simplified. In this case we note indeed that $f(\delta_r, L_r)$ is almost constant:

$$f(\delta_r, L_r) \approx \frac{4L_r}{\pi}$$  \hspace{1cm} (9.20)

As a result, $\delta_p$ which now assumes the following expression:

$$\delta_p = 1 - N_{L_p}\left(\frac{4L_r}{\pi\omega}\right)$$  \hspace{1cm} (9.21)

do es no longer depend on $\delta_r$, but only on the pulsation $\omega$.

**Variation on the $\mu$ lower-bound computation**

When computing the $\mu$ lower-bound, no relationship between $\delta_p$ and $\delta_r$ is a priori taken into account. Yet, it appears that the method we have used can be easily adapted. As a matter of fact, when the imaginary axis is reached for an admissible perturbation $\Delta_i^* = \text{diag}(\delta_p^*, \delta_r^*)$ at some frequency $\omega_i^*$, further iterations can be performed along the imaginary axis. As already mentioned, this can be done so as to further minimize the norm of $\Delta$. But, it can also be done to change the frequency for which the condition of oscillation is verified.
As illustrated by figure (9.6), for each previously determined critical frequency point \( \omega_i^* \), we now obtain a frequency interval \([\omega_i, \omega_f]\) associated to a continuous set \( \Delta_i(\omega) \). The limits of the interval correspond to values of \( \omega \) for which the elements of \( \Delta_i(\omega) \) do no longer belong to \([0, 1]\).

Since we now have two functions \( \delta_{ip}(\omega) \) and \( \delta_{ir}(\omega) \) instead of two single points, the equation (9.19) can be taken into account. It suffices indeed to check if the following equation:

\[
\delta_{ip}(\omega) + N_{L_p} \left( \frac{f(\delta_{ir}(\omega), L_r)}{\omega} \right) = 1
\]

(9.22)

admits solutions on the interval \([\omega_i, \omega_f]\). If no solution can be found, then no limit-cycle will appear. If a solution (or several) is found at pulsation \( \omega \), then a limit-cycle may occur at this pulsation. Its amplitude can be determined via the associated saturation degrees. It is also easily possible to characterize its stability following the method detailed in [And98].

**Important remark**

If the position limiter is placed before the rate limiter, the relationship between \( \delta_p, \delta_r \) and \( \omega \) will be different. It is easily shown that, in this case:

\[
N_{L_p}(x_p) x_p = \left( \tau - j \frac{N_{L_r}(x_r)}{\omega} \right) x_r
\]

(9.23)

Thus, for a given rate saturation level \( \delta_r \), \( \delta_p \) can be easily computed in a similar way as in (9.22). Furthermore, when both saturation are rather active, (which is generally the case when P1O is fully developed), the equation (9.23) simplifies to:

\[
\tau x_r = \frac{4}{\pi \tau} \sqrt{\frac{L_p^2}{\omega^2} - \frac{L_r^2}{\omega^2}}
\]

(9.24)

which shows that any limit-cycle in this case will occur at a frequency which is greater than \( \frac{L_r}{L_p} \). This is of course only true if the first harmonic approximation is valid.

The equation (9.19) is thus replaced by:

\[
\delta_p = f^{-1} \left( \frac{f(\delta, L_r)}{\omega}, L_p \right)
\]

(9.25)
and equation now becomes:

$$\delta_p(\omega) - f^{-1}\left( \frac{f(\delta_p(\omega), L_p)}{\omega}, L_p \right) = 0 \quad (9.26)$$

### 9.4 Interpretation of the $\mu$ upper-bound

The computation of the $\mu$ upper-bound is of high interest in robustness analysis and synthesis. It provides indeed a guaranteed robustness margin. The conservatism of this margin is directly linked to the efficiency of the algorithms.

#### 9.4.1 A reliable approach

Several results are available to compute a $\mu$ upper-bound. We recall hereafter the classical mixed-$\mu$ upper-bound proposed by [FTD91].

$$\mu_\Delta(M) \leq \left[ \max(0, \inf_{D \in \mathcal{D}} \inf_{G \in \mathcal{G}} \chi(M^*DM + j(GM - M^*G), D)) \right] \quad (9.27)$$

where $D \in \mathcal{D}$ and $G \in \mathcal{G}$ are scaling matrices whose structure is deduced from the structure of the model perturbation $\Delta$. Note that this upper-bound can be computed using the LMI optimization tools provided in the software package $LMI - lab$ for use with Matlab.

In the case of purely real $\mu$ problems, some difficulties may appear due to possible discontinuities of the exact value of $\mu$ with respect to frequency. Therefore, using the above result to compute a $\mu$ upper-bound at each point of a frequency gridding is not safe. A reliable approach was proposed in [FB88]. It consists of computing $D$ and $G$ scaling, simultaneously “working” at two neighboring frequencies, i.e., verifying (for a minimized $\beta$):

$$M(j\omega_1)^*DM(j\omega_1) + j(GM(j\omega_1) - M^*G(j\omega_1)) < \beta^2D \quad (9.28)$$

$$M(j\omega_2)^*DM(j\omega_2) + j(GM(j\omega_2) - M^*G(j\omega_2)) < \beta^2D \quad (9.29)$$

Then, the validity of the scalings is checked on the whole interval, using a simple test which only involves eigenvalues determination. If the test fails, then the interval $[\omega_1, \omega_2]$ is further split.

#### 9.4.2 Interest for limit-cycle analysis

As is clear from above, the computation of a $\mu$ upper-bound does not involve the computation of a perturbation, since it is based on the determination of scaling matrices. However, it may reveal interesting in the framework of non-linear analysis, since it can be used to determine frequency intervals where a limit-cycle is very unlikely to occur.

Assume indeed that a $\mu$ upper-bound has been computed on the frequency interval $[\omega_1, \omega_2]$ for our particular non-linear problem which involves a diagonal real perturbation. Suppose
then that:

\[ \forall \omega \in [\omega_1, \omega_2], \quad \mu_{\Delta}(L(j\omega)) < 1 \]  

(9.30)

We deduce that on this interval, the necessary condition of oscillation (9.7) can only be verified for \( \Delta(x(\omega)) \) such that:

\[ \sigma(\Delta(x(\omega))) > 1 \]  

(9.31)

which is a non-admissible perturbation. Consequently, provided of course that the first harmonic approximation is valid\(^1\), we can conclude that no limit-cycle will be observed on the interval \([\omega_1, \omega_2]\).

### 9.5 A proposed PIO criterion

With the help of the above results - mainly concerning \( \mu \) lower-bound computation - we now propose and justify a PIO criterion. For given rate and position limits, the objective is to predict whether PIO is likely to appear or not.

It should be remarked that the potential existence of a limit-cycle is not directly correlated to a PIO occurrence. This will highly depend on the pilot task. But, on the contrary, if no limit-cycle was detected, then Cat-II PIO will not occur, whatever the pilot task. This case is illustrated by figure (9.7). The two continuous-line plots represents \( \delta_p \) and \( \delta_r \) versus the frequency. The dashed line which does not intersect \( \delta_p \) represents the function \( 1 - N_{Lp}(f(\delta_r(\omega))/\omega) \). It results that the equation (9.22) cannot be verified.

![Graph](https://placehold.it/300x300)

**Fig.9.7 Absence of limit-cycle**

We now focus on the case where at least one limit-cycle has been detected via the modified \( \mu \) lower-bound computation approach. This situation is illustrated by figure (9.8).

\(^1\)We recall indeed that the necessary condition of oscillation involving a diagonal real-valued operator \( \Delta(x(\omega)) \) relies on the first harmonic approximation. If super-harmonic part of the signals are present, then \( \Delta \) becomes complex and a complex \( \mu \) upper-bound has to be considered. Expected results would in this case be rather conservative.
If both limit-cycles are unstable, then they will not severely affect the behavior of the system and there is no PIO risk. Now, assume that one limit-cycle at least is stable. The problem is to know whether or not, during a specific pilot-task, the critical saturation degrees will be reached.

To answer this question, we will compute the frequency at which the rate limiter is activated for the first time under closed-loop condition and maximum pilot input amplitude. This frequency is denoted $\tilde{\omega}_{onset}$ and referred to as the closed-loop onset frequency. Its computation is detailed in [Dud97c].

9.5.1 A graphical approach

The next two figures (9.9)-(9.10) illustrate the proposed PIO criterion, which is essentially graphical. These figures are basically the same as figure (9.8). A vertical line has been added, which simply corresponds to the onset frequency. What then permits to evaluate the PIO risk is the relative position of this line and the unstable limit cycle. On figure (9.9), a high PIO risk is illustrated, whereas on figure (9.10) the risk is low.

---

**Fig.9.8 Illustration of limit-cycle detection**

**Fig.9.9 Illustration of a high PIO risk**
Fig.9.10 Illustration of a low PIO risk

For the configuration considered here, the criterion could be:

$$\hat{\omega}_{\text{onset}} < \omega_2$$  \hspace{0.5cm} (9.32)

Note that if the unstable limit-cycle turns out to be stable, the criterion becomes:

$$\hat{\omega}_{\text{onset}} < \omega_{\max}$$  \hspace{0.5cm} (9.33)

To justify this criterion, the simplest way is to consider the Nyquist locus (see figure (9.11)), on which we have drawn the critical locus (negative inverse describing function), as is usually in first harmonic analysis method. In the case of a single rate saturation, the critical locus is a vertical line. We note here two intersections at $\hat{\omega}_1$ and $\hat{\omega}_2$ which are close to $\omega_1$ and $\omega_2$ if the rate limiter is much more activated than the position limiter (which is not necessarily the case).

Fig.9.11 Interpretation with Nyquist locus

On this figure, it is clear that the intersection at $\hat{\omega}_1$ corresponds to stable limit cycle whereas, the second intersection corresponds to an unstable limit-cycle. Therefore, if the onset frequency verifies:

$$\hat{\omega}_{\text{onset}} > \hat{\omega}_2$$ \hspace{0.5cm} (9.34)

the PIO risk is low. Otherwise the risk is high. This criterion, is similar to the one proposed above. The main difference is that our proposed criterion accounts for simultaneous position and rate limiters. Since $\hat{\omega}_2 \geq \omega_2$, our criterion may predict a PIO risk, whereas an approach only based on rate saturation effects would not detect any PIO tendency.
9.5.2 Reliability improvement

It should be pointed out that the proposed PIO criterion may be pessimistic is some cases for the following two reasons, which both are related to the onset closed-loop frequency:

- the determination of this frequency is based on a constant pilot input signal, of maximum amplitude, which may not be realistic. It is meant there that this worst case signal is overly stringent, which leads to a low (then pessimistic) estimation of the onset frequency. As a result, PIO might be predicted whereas it does not appear during a real task,

- conservatism is also introduced when comparing the onset frequency with the frequency of the limit-cycle. The first frequency is associated to the first apparition of rate saturation ($\delta_r \approx 0$) whereas in the second case, the saturation might be much more effective. In a more rigorous approach it would be desirable to determined a modified onset frequency (which is higher than the original one) associated to the saturation degree of the rate limiter. However, this computation is difficult since after first occurrence of rate limiting, the system is no longer linear... It should also be remarked that, once it has first appeared, saturation effect generally grows rapidly. As a result the conservatism which is introduced there is expected to be low.

From the above remarks, reliability improvements of our PIO criterion can clearly be achieved. But it required more computational efforts. Moreover, our criterion is already reliable in a sense. As a matter of fact, when no PIO is detected, then this information is guaranteed to be true. Now, if PIO is detected, then we can conclude that the PIO risk is high, without guaranteeing that it will indeed occur during a flight.

9.6 Conclusion and remarks

In this section we have presented a detailed description of the applied analysis technique. The interests of $\mu$ tools for limit-cycle detection have been highlighted. It was especially shown that the computation of $\mu$ lower-bounds offers many possibilities of solving the necessary condition of oscillation. Some adaptation of the original algorithm to tackle this problem has been presented. Finally, a PIO criterion was derived.

To conclude the section, we now propose to take a look at the main limitations and possible extensions of the method.

9.6.1 Current main limitations

**Numerical difficulties**

The limit-cycle detection phase is initiated by a $\mu$-lower bound search, which is known to be a non-convex problem. Our algorithm usually performs well, but some difficulties may
appear for badly chosen initial point \((\delta_{0}, \delta_{r0})\). Since the approach is computationnaly cheap, several trials can be made, until a good solution is obtained.

The computation of a \(\mu\) upper-bound is a convex problem. So, technical difficulties are avoided here, but it appears to be rather time consuming if a high precision level is required (dense frequency gridding). Fortunately this bound is not absolutely necessary in our technique. It can just be used a posteriori, to check that a frequency interval is free of any potential limit-cycle.

**Non-linearity type**

An important limitation of this method at present stage of its development is the restricted class of non-linearities that can be treated. An important fact indeed in our approach is the realness of the equivalent gain. Therefore, the non-linearity needs to be symmetric and defined by a single plot. Non-linearities such as hysteresis which is defined by multiple plots cannot be treated. Currently our approach is then limited to cat. II PIO analysis. Yet, work is currently carried out to study the possibilities of extensions to the case of complex gains.

**Detection of critical pilot inputs**

The proposed PIO criterion is based on a worst case pilot input which is simply defined as a constant input of maximum amplitude. This type of input is of course not always realistic. As a result, in some given configuration, the method may predict a PIO tendency whereas no PIO will occur during a realistic simulation. The criterion would then be less conservative if a realistic pilot input could be entered to determine the onset frequency point. It is believed that a specific \(\mu\)-type analysis could compute such signals, but as far as we know, such analysis techniques have not been implemented and tested yet.

9.6.2 Short-term possible extensions

**Multi-channel analysis**

In our description of the analysis technique we have focused on the special case of simultaneous position and rate limiters, on a single channel. In most cases, this type of analysis gives satisfactory results since - even in the case of multi-inputs systems - limitations generally occur first on a single channel. Yet, in some specific cases, such as highly coupled systems, results could be interestingly improved by accounting for possibly simultaneous position and rate limiting on several inputs. In our proposed framework, this extension to multi-channels non-linear analysis can be performed in a short-term. The main work consists of adapting the last phase of our modified \(\mu\) lower-bound computing technique.

**Uncertainties**

Since our method relies on an LFT framework, accounting for parametric uncertainties can be viewed as a natural extension. It is known that a PIO free system can rapidly loose this property even for small parametric variations (such as the center of gravity location
for example). Here a fairly standard $\mu$ analysis can be applied to evaluate the robustness of the “PIO free property” versus parametric variations. The approach can be shortly describe as follows (see [FF98]).

- Form the LFT system including a block associated to the uncertain parameters,
- Define a frequency gridding $W$ around the pulsation where the robustness is to be evaluated,
- Define a frequency dependent gridding $X$ for the admissible values of the describing functions values,
- At each frequency point, compute $\mu$ upper-bounds on the frequency-depend gridding
- The robustness margin is obtained as:

$$ k = \frac{1}{\max_{w,x} \bar{\mu}_{\Delta_{\text{par}}}(\omega, x(\omega))} $$

An alternative method based on rational fitting of the describing function and using a generalized $\mu$ approach is proposed in [Tie97].
10 Description of the analysis cycle

For a selected configuration that is:

- the pilot gain is fixed $K_{pil}(\phi_c)$,
- the position and rate saturation levels are given

we describe in this section the main steps which are necessary to complete the analysis. They will be illustrated on the F-18, for a cross-over phase angle $\phi_c = -130^\circ$ a medium rate saturation level $L_r = 140 \text{ deg/s}$, and a position saturation level given by $L_p = 40 \text{ deg}$.

10.1 Step 1: Onset frequency computation

A detailed description of this first step is not necessary here since the procedure is exactly the same as the one followed in [Dud97c, DD99]). For the selected configuration we have found:

$$\omega_{onset} = 2.97 \text{ rad/sec}$$

10.2 Step 2: generation of the LFT

This step is essential in our approach. It can be done very easily in Simulink environment as shown on following figures, for the F-18 aircraft, with position and rate saturation on one single input. Figure (10.1) gives a general view of the LFT. Figures (10.2) clearly shows how position and rate saturations have been introduced in the analysis. Figure (10.3) presents an alternative solution if the position limiter is placed before the rate limiter.

![Fig.10.1 LFT simulink diagram of F-18](image-url)
10.3 Step 3: limit-cycle(s) research

This is the main step. It can be further divided into three tasks which consists of:

- solving the necessary condition of oscillation,
- detecting potential limit-cycle
- applying the PIO criterion

10.3.1 Solving the necessary condition of oscillation

This phase is the most time-consuming in our approach. On a Sun Ultra 5 workstation (270 MHz), it takes about one minute for the F-18 example. It has to be repeated for each different pilot gain, but it is important to note that this step is independent of saturations levels.

Two Matlab functions were developed to solve the equation. As explained in previous section, the method is based on a modified $\mu$-lower bound computation algorithm. The first Matlab function called `mulb.m` performs a rough $\mu$ lower-bound computation, using the Frobenius norm. It provides initial inputs to the second function called `musat.m` which is devoted to the final optimization step along the imaginary axis. The outputs of this function is a frequency range and corresponding saturation degrees where the necessary condition of oscillation can be verified.

Let us call `LFT_f18.mdl` the LFT Simulink model defined in step 2. We now present below the Matlab command lines permitting to compute the frequency range ($\Omega_m$) and the saturation degrees ($D_p$ and $D_r$). Note that all this variables are vectors. The variable named `Stab` is a stability indicator of the potential limit-cycle.

Here comes the verbatim text:

In this example, the function `mulb.m` only found one admissible solution. So the size of output `delta` is $2 \times 1$. If several solutions were found, then `mustab.m` would have been
called as many times as the number of solutions.

The results are shown on the following two figures. Figure (10.4) presents the position saturation degree w.r.t. pulsation. In dashed-line, the stability indicator is presented. We note that at low frequency the potential limit-cycle is stable. For frequency above 0.86 rad/sec it is unstable.

![Graph of Position Saturation Degree](image1)

**Fig.10.4 Position saturation degree: \( \delta_p(\omega) \)**

![Graph of Rate Saturation Degree](image2)

**Fig.10.5 Rate saturation degree: \( \delta_r(\omega) \)**

10.3.2 Limit cycle detection

This step now depends on the saturations level. It consists indeed of checking whether the equation (9.22) admits solutions or not. To proceed, we simply plot on figure (10.4) the
function:

\[ 1 - N_{L_p} \left( \frac{f(\delta_p(\omega))}{\omega} \right) \]

If the position saturation is placed before the rate saturation, then the amplitude saturation level \( \delta_p \) is calculated as a function of \( \delta_r \) and the pulsation, by applying equation 9.23.

On the figure below (figure (10.6)) we present the standard case, when the rate saturation occurs first. Here, two limit cycles are detected. The first one (low frequency) is stable. The second one (high frequency) is unstable.

![Limit-cycle detection](image)

**Fig.10.6 Limit-cycle detection**

### 10.3.3 Application of PIO criterion

To apply the PIO criterion it now only remains to place the vertical line associated to the onset frequency (see figure (10.7)).
Fig. 10.7 Application of PIO criterion

Here, it appears that the PIO risk is very high.
11 Results of the analysis technique on F-18

We now present the results that were obtained on F-18 for different configurations:

- $\phi_c = -110^\circ, -120^\circ, -130^\circ, -140^\circ, -150^\circ, -160^\circ$
- $L_r = 80 \text{deg/sec}, L_r = 140 \text{deg/sec}, L_r = 200 \text{deg/sec}$

Moreover, two cases are considered according to the relative positions of rate and position limiters:

- **First case**: Rate limiter before position limiter
- **Second case**: Position limiter before rate limiter

11.1 Summary of the results

11.1.1 Case I

The interpretation of the figures (see below) leads to the following results which are summarized in a table. The stars ($^*$) indicate some differences with respect to OLOP criterion. The symbol ($\#$) indicates differences with respect to the time-domain simulation which was performed for a constant pilot input of maximum amplitude ($25 \text{deg}$) during $5 \text{sec}$ (see figure (11.13). It appears that our criterion matches very well with the time simulations. The case which is indicated by ($\#$) is ambiguous. The criterion predicts PIO and the time-domain simulation shows a stable behavior at last but after several poorly damped oscillations.

<table>
<thead>
<tr>
<th>$\phi_c$</th>
<th>$L_r = 80$</th>
<th>$L_r = 140$</th>
<th>$L_r = 200$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-110^\circ$</td>
<td>NO PIO</td>
<td>NO PIO (*)</td>
<td>NO PIO</td>
</tr>
<tr>
<td>$-120^\circ$</td>
<td>PIO (#)</td>
<td>NO PIO (*)</td>
<td>NO PIO</td>
</tr>
<tr>
<td>$-130^\circ$</td>
<td>PIO</td>
<td>PIO</td>
<td>NO PIO</td>
</tr>
<tr>
<td>$-140^\circ$</td>
<td>PIO</td>
<td>PIO</td>
<td>PIO (*)</td>
</tr>
<tr>
<td>$-150^\circ$</td>
<td>PIO</td>
<td>PIO</td>
<td>PIO (*)</td>
</tr>
<tr>
<td>$-160^\circ$</td>
<td>PIO</td>
<td>PIO</td>
<td>PIO</td>
</tr>
</tbody>
</table>

*Table 11.1 Summary of the results*

11.1.2 Case II

In this case, a new problem occurs, which is due to a lack of precision of the first harmonic approximation. Here indeed the saturation non-linearities are no longer separated by a
low-pass linear system. If we compare figures (11.7-11.12) with time-responses (11.14), we note indeed important discrepancies between pulsations of predicted limit-cycles and actual ones. As a result, the application of the PIO criterion is not reliable in this case.
11.2 Plots associated to the PIO criterion: Case I

Fig. 11.1 $\phi_c = -110^\circ$ - Case 1

Fig. 11.2 $\phi_c = -120^\circ$ - Case 1
Fig. 11.3 $\phi_c = -130^\circ$ - Case 1

Fig. 11.4 $\phi_c = -140^\circ$ - Case 1
Fig.11.5 $\phi_e = -150^\circ$ - Case 1

Fig.11.6 $\phi_e = -160^\circ$ - Case 1
11.3 Plots associated to the PIO criterion: Case II

Fig. 11.7 $\phi_c = -110^\circ$ - Case 2

Fig. 11.8 $\phi_c = -120^\circ$ - Case 2
Fig. 11.9 $\phi_k = -130^\circ$ - Case 2

Fig. 11.10 $\phi_k = -140^\circ$ - Case 2
Fig.11.11 $\phi_c = -150^\circ$ - Case 2

Fig.11.12 $\phi_c = -160^\circ$ - Case 2
11.4 Time-domain plots

Fig. 11.13 Time response to a square input: Conf 1
Fig.11.14  Time response to a square input: Conf 2
12 Conclusions, Part II

In this report we have proposed a new extended Cat II. PIO criterion which performs well in the presence of simultaneous amplitude and rate saturations. The method is checked on various configurations of F-18 lateral system. On this system, saturations are mainly active on one input. However, it is interesting to point out that the proposed technique could be extended to handle more general cases, when saturations simultaneously appear on several inputs.

Our method gave good results - rather similar to OLOP criterion - when the rate saturation is placed before the amplitude saturation. When the latter is placed before, some problems occur which are related to the validity of the first harmonic approximation. The basic assumption regarding the low-pass behavior of linear parts of the system is no longer verified. In this case, extension possibilities to handle higher harmonic parts of the signals have to be studied. What is interesting in our approach is that the $\mu$-type plots, provided that equivalent gains remain real, are still valid.

For shortness, results regarding the application of the $\mu$ upper-bound have not been presented. For various configuration, it has been observed that the $\mu$ upper-bound plots became smaller that one for frequencies above 6 to 7.5 rad/s. This indicates that no limit-cycle can be observed for higher frequencies. If the latter had been smaller that the onset frequency, we could have announced that the system was PIO-free. But, this criterion is quite conservative.

Finally, although a few weak points still require some further investigation, it can be outlined that the LFT formalism associated to $\mu$ tools (with some adaptations) provide an interesting framework for PIO analysis.
PART III

System analysis using Integral Quadratic Constraints
13 Description of the analysis technique

Integral Quadratic Constraints (IQC) provide a way of representing relationships between processes evolving in a complex dynamical system, in a form that is convenient for analysis. In this section, a short introduction is given on IQC theory. For more profound details on IQC theory, the reader is referred to [MR97], [RM97], [SW99].

13.1 Notation and Terminology

Here $A^*$ denotes the complex conjugate transpose, $\hat{v}(j\omega)$ represents the Fourier transform of $v(t)$. $RL_\infty$ denotes the set of proper (bounded at infinity) rational functions with real coefficients. The subset consisting of functions that are Hurwitz is denoted $RH_\infty$. $L_2[0, \infty)$ can be thought of as the space of real-valued functions $f : [0, \infty) \to R$ with finite energy

$$\|f\|^2 = \int_0^\infty |f(t)|^2 dt < \infty.$$ 

This is a subset of the extended space $L_{2e}[0, \infty)$, whose members only need to be square integrable on finite intervals. By an operator we mean a function $F$ from one $L_2[0, \infty)$ space to another. The $L_2$-gain of an operator $F : L_2[0, \infty) \to L_2[0, \infty)$ is given by

$$\|F\| = \sup \|F(f)\| / \|f\| : f \in L_2[0, \infty), f \neq 0.$$ 

13.2 Preliminaries

Integral Quadratic Constraints provide a simple and efficient way of analyzing stability and performance of feedback interconnections of the form shown in figure 13.1.

![IQC Analysis Setup](image)

The relation between the various signals in the figure are described as follows:

$$v = Gw + f$$
$$w = \Delta v + e$$

(13.1)
matrix $G(s)$, and a nonlinear, time-varying, and possibly uncertain operator $\Delta$. Since we assume $G(s)$ to be linear, we confine ourselves to systems that have isolated nonlinearities that can be ‘pulled out’ into the structure of figure 13.1. Here $G \in RH_\infty$ is a stable system and $\Delta$ is a bounded causal operator. An operator $\Delta$ is causal if the past of the output does not depend on the future of the input. A causal operator is bounded if there exists $c > 0$ such that $\|\Delta(v)\| < c\|v\|$, for all $v \in L_2$. The lower bound of such constants is called the gain of $\Delta$. In figure 13.1 the signal $e \in L_{2\infty}[0, \infty)$ plays the role of interconnection noise and the signal $f \in L_{2\infty}[0, \infty)$ is the external disturbance. $(v,w)^T$ constitutes the response of the interconnection. We say that the interconnection of $G$ and $\Delta$ is well-posed if the map $(v,w) \rightarrow (e,f)$ defined by eq. (13.1) has a causal inverse on $L_{2\infty}[0, \infty)$. The interconnection is stable if, in addition, the inverse is bounded.

In the second step, $\Delta$ is described in terms of IQCs, which means inequalities of the form [RM97]

$$\int_{-\infty}^{\infty} \left[ \frac{\hat{v}(j\omega)}{\Delta v(j\omega)} \right]^* \Pi(j\omega) \left[ \frac{\hat{v}(j\omega)}{\Delta v(j\omega)} \right] d\omega \geq 0, \quad \forall w = \Delta v \quad (13.2)$$

relating the input and output of $\Delta$, under the assumption that both are square and integrable. Two signals $\Delta v \in L_2[0, \infty)$ and $v \in L_2[0, \infty)$ are said to satisfy the IQC defined by $\Pi$ if eq. (13.2) holds. Here the Fourier transforms $\hat{v}(j\omega)$ and $\Delta v(j\omega)$ represent the harmonic spectrum of the signals $v$ and $\Delta v$ at the frequency $\omega$, and eq. (13.2) describes the energy distribution in the spectrum of $(v, \Delta v)$. $\Pi$ is a bounded operator and is called a multiplier. In many cases, such IQCs are readily available in literature, though usually not in explicit form. As a rule, IQCs for $\Delta$ are produced by forming any convex combination of ‘standard’ IQCs derived for elementary subsystems of $\Delta$.

In the third step, a matrix function $\Pi$ is sought amongst all multipliers satisfying eq. (13.2), that also satisfies the condition:

$$\left[ \begin{array}{c} G(j\omega) \\ I \end{array} \right]^* \Pi(j\omega) \left[ \begin{array}{c} G(j\omega) \\ I \end{array} \right] \leq -\epsilon I, \quad \forall \omega \in \mathbb{R} \quad (13.3)$$

with $\epsilon > 0$. This search amongst $\Pi$ can be expressed in terms of solvability of a system of Linear Matrix Inequalities (LMIs). If a solution exists, then the feedback system is stable, provided that certain general non-restrictive assumptions are satisfied by $\Pi$, $\Delta$ and $G$. The obtained result is summarized in the following theorem:

**Theorem 13.1** Incremental $L_2$-gain theorem. [MR97] Let $G(s) \in RH_\infty$ and let $\Delta$ be a causal operator. Assume that

i). For every $\tau \in [0,1]$, the interconnection of $G$ and $\tau \Delta$ is well-posed.
ii). For every $\tau \in [0,1]$, $\tau \Delta$ satisfies the IQC defined by $\Pi$ according to eq. (13.2).

iii). There exists $\epsilon > 0$ such that eq. (13.3) holds.

Then the feedback interconnection of $G$ and $\Delta$ is stable.

Theorem 13.1 can be interpreted as follows: for a specific uncertainty block, find as many multipliers that all comply with equation 13.2. If for any one $\Pi$ equation 13.3 holds, then it can be concluded that the mapping $(\epsilon, f) \rightarrow (u, w)$ is stable. If $\tau \Delta$ satisfies several IQCs, defined by $\Pi_1, \ldots, \Pi_n$, then a sufficient condition for stability is the existence of $x_1, \ldots, x_n \geq 0$ such that eq. (13.3) holds for $\Pi = x_1 \Pi_1 + \ldots + x_n \Pi_n$. Hence, out of a finite number of IQCs, one can construct an infinite set of multipliers. The more multipliers one can find, the better, since for only one multiplier eq. (13.3) needs to hold in order to prove stability.

There is a close relationship between quadratic stability and stability analysis based on IQCs. As a rule, if a system is quadratically stable, then its stability can also be proved by using a simple IQC. Conversely, a system that can be proved to be stable via IQCs always has a quadratic Lyapunov function in some generalized sense. In any case, use of IQCs replaces the "blind" search for a quadratic Lyapunov function, which is typical for quadratic stability, by a more sophisticated search [MR97].

One of the most appealing features of IQCs is their ability to widen the field of application of already existing results. This means that almost any robustness result derived by some method (possibly unrelated to the IQC techniques) for a special class of systems can be translated into an IQC.

### 13.3 IQCs for Critical Operators

The multipliers relevant for our study are given next. The proofs for the specific IQCs can be found in the references [MR97, Meg97]

**IQC for uncertain LTI dynamics** [MR97]

Let $\Delta$ be any linear time invariant operator with $H_\infty$ norm less than one. Then $\Delta$ satisfies all IQCs of the form

\[
\Pi_U(j\omega) = \begin{bmatrix} x(j\omega)I & 0 \\ 0 & -x(j\omega)I \end{bmatrix}
\]

where $x(j\omega) \geq 0$ is a bounded measurable function.

**IQC for time varying real scalar** [MR97]
Let $\Delta$ be defined by multiplication in the time domain with a scalar function $\delta \in L_\infty$ with $\|\delta\|_\infty \leq d$. Then $\Delta$ satisfies the IQC defined by $\Pi$, 

$$
\Pi_{TV} = \begin{bmatrix}
    d^2X & Y \\
    Y^T & -X 
\end{bmatrix}
$$

(13.5)

where $X = X^T \geq 0$, and $Y = -Y^T$ are real matrices.

**IQC for rate limiting elements [Meg97]**

The IQCs that have been derived for rate limiting elements so far, are based on an approximation of the nonlinear dynamics by incorporating a saturation element $sat(e)$ and an integrator in a feedback loop, see figure 13.2. A saturation operator $sat(e)$ is defined as:

$$
sat(e) = \frac{e}{\max\{1,|e|\}}
$$

where $sat$ satisfies $e \cdot sat(e) \geq 0$. The output $z(t)$ in figure 13.2 approximates the effect of an exact rate limiter with rate $R$ to a commanded input $r(t)$. This approximation becomes more exact as the gain $K_{sat} > 0$ is increased. We will choose $K_{sat} = 1$ for the purpose of this study.

![Rate Limiter Diagram](image-url)

*Fig. 13.2 Rate limiter approximation used for IQC analysis*

The inner operator from $u(t)$ to $y(t)$ can be described in terms of IQCs. For technical reasons we will not concern ourselves with, the approximation scheme is extended with a $(s+1)$ block, and the IQCs apply to the mapping of $u(t)$ to $y(t)$. In order to get back to the approximation an additional $\frac{1}{s+1}$ block is needed, see figure 13.2. Define $\Delta_{RL}$ as the
operator \( u \rightarrow y \), where
\[
\begin{align*}
\dot{x} &= \text{sat}(u - x), \quad x(0) = 0 \\
y &= x + \text{sat}(u - x)
\end{align*}
\]
This operator is stable, and the input \( u \) and the output \( y \) satisfy the following IQCs:

- **Sector IQC for rate limiter**
  \[
  \Pi_{\text{sector}}(j\omega) = \begin{bmatrix}
  0 & \frac{j\omega}{1-j\omega^2}x \\
  \frac{-j\omega}{1-j\omega^2}x & \frac{-\omega^2}{\omega^2+1}x
  \end{bmatrix}, \quad x > 0
  \tag{13.6}
  \]

- **Popov IQC for rate limiter**
  \[
  \Pi_{\text{popov}}(j\omega) = \pm \begin{bmatrix}
  0 & \frac{-\omega^2}{(1+j\omega)^2}x \\
  \frac{-\omega^2}{(1-j\omega)^2}x & \frac{-\omega^2}{\omega^2+1}x
  \end{bmatrix}, \quad x > 0
  \tag{13.7}
  \]

- **Zames and Falb IQC for rate limiter**
  \[
  \Pi_{\text{zames-falb}}(j\omega) = \begin{bmatrix}
  0 & \frac{j\omega(1+H)}{j\omega+1}x \\
  -\frac{j\omega(1+H^*)}{j\omega+1}x & \left( \frac{j\omega(1+H^*)}{j\omega+1} - \frac{-\omega^2}{\omega^2+1} \right) x
  \end{bmatrix}
  \tag{13.8}
  \]
where \( x > 0 \), and \( H = H(j\omega) \in RL_\infty \) has the property that its impulse response \( h(t) \) satisfies \( \int_{-\infty}^{\infty} |h(t)| dt \leq 1 \).

- **Sasha IQC for rate limiter**
  \[
  \Pi_{\text{sasha}}(j\omega) = H(j\omega)^+ \begin{bmatrix}
  \sigma_{11} & \sigma_{12} & \sigma_{13} \\
  \sigma_{12} & \sigma_{22} & \sigma_{23} \\
  \sigma_{13} & \sigma_{23} & \sigma_{33}
  \end{bmatrix} H(j\omega)
  \tag{13.9}
  \]
where
\[
H(j\omega) = \begin{bmatrix}
  0 & \frac{1}{j\omega+a} \\
  \frac{-1}{j\omega+a} & 1 \\
  \frac{j\omega}{j\omega+a} & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
  \sigma_{11} & \sigma_{12} \\
  \sigma_{12} & 2\sigma_{22}
\end{bmatrix} \geq 0, \quad \sigma_{22} + 2\sigma_{23} + \sigma_{33} \geq 0, \quad \sigma_{33} \leq 0.
\]

We can combine multiple IQCs by taking their convex hull. In this case, we can describe the relation between in- and output of the rate limiting approximation by:
\[
\Pi_{RL} = \sum_{j=1}^{k} \tau_j \Pi_j, \quad \tau_j \geq 0
  \tag{13.10}
\]
where \( \Pi_j \) represents any of the IQCs (13.6) to (13.9), of which some can be parameterized according to their definitions.
13.4 Example application of IQC analysis

An IQC-based approach to robust stability performance evaluation is relatively new. Successful applications have only been emerging only over the last couple of years. However, the great potential of the methods has been foreseen by researchers of the Massachusetts Institute of Technology (US) and Lund Institute of Technology (Sweden). In a joint effort, a Matlab/Simulink-based toolbox for IQC analysis, called IQCβ, has been written [MKJR99]. This toolbox simplifies implemental and computational issues of IQC analysis to a great extent. A considerable class of uncertain blocks are described in a standard way such that the accompanying IQCs can be used.

In this section, an example case will be presented to demonstrate the IQC analysis described above. The IQCβ toolbox will be used, and a very concise description of its functionality is offered. We will compare IQC-based approach to the stability analysis with a sinusoidal describing function approach, which can be considered to be a classical approach. The system we will consider is a simple SISO feedback system shown in 13.3. We will choose:

\[ G_p(s) = \frac{K_{num}}{s^2 + 2 \times 1.73 \times 0.58s + 1.73^2} \]

The output \( w(t) \) of a second order system \( G_p(s) \) is fed back, and the rate limited error signal serves as the input to \( G_p \). In the linear case, i.e. when no rate limiter is present, it is obvious that the closed-loop system remains stable for all \( K_{num} > 0 \). With the rate limiter in place, the system can be susceptible to limit cycles. These are sustained, nonlinear oscillations with fixed amplitude and frequency to which the system response converges if it is excited to a certain extent, even if the input decays to zero after some time. [GM61] gives a thorough description of this phenomenon as well as a method to detect the existence of limit cycles. In this example, we will compare this method (which uses the sinusoidal describing function technique) with an IQC-based approach.

![Fig.13.3 SISO feedback loop containing a rate limiter](image-url)
13.4.1 Analysis using sinusoidal describing functions

The conditions for a limit cycle can be sought using describing function analysis. The describing function technique is a convenient way of dealing with single isolated nonlinearities. Essentially, it approximates input and output signals of the nonlinearity as sinusoids and defines amplitude ratios and phase shifts that depend on frequency and amplitude of the input signal. This will allow for the use of the same techniques that are used for linear systems (i.e. the describing function is the nonlinear equivalent to the linear transfer function). In order to calculate the describing function, the Fourier Transform is applied to the input and output signals, and these series are broken off using only the ground frequency terms. The error that is made by doing so is compensated for by adding a remnant signal to the describing function. The remnant is a high frequency signal, which can be neglected if the linear part of the system has low pass character.

[Dud97a] derives a describing function for a fully saturated rate limiting element; the describing function \( N(j\omega, \hat{r}) \) is a function of both amplitude \( \hat{r} \) and frequency \( \omega \) of the sinusoidal input signal \( r \) to the rate limiter.

\[
N(j\omega, \hat{r}) = \frac{4}{\pi} \frac{\omega_{onset}}{\omega} e^{-j \arccos(\frac{\omega_{onset}}{\omega})}, \quad \text{with:} \quad \omega_{onset} = \frac{R}{\hat{r}}
\]  

(13.11)

Using a simple SISO Nyquist argument, neutral stability of the closed loop system is achieved when the loop transmission approaches the point \(-1\), i.e.:

\[
N(j\omega, \hat{r}) \cdot G_P(j\omega) = -1, \quad \Rightarrow \quad G_P(j\omega) = \frac{-1}{N(j\omega, \hat{r})}
\]  

(13.12)

![Nichols Plot](image)

*Fig. 13.4 Nichols plot of \(-1/N(j\omega, \hat{r})\) and \(G_P(j\omega)\) revealing limit cycle conditions*
This latter equality can be checked graphically by plotting $G_P$ and $-1/N$. Such a plot (in this case, a Nichols diagram) is shown in figure 13.4. The frequency response $G_P(j\omega)$ is plotted for different values of the steady-state gain $K_{num}$. From this figure, it can be concluded that for $K_{num} > 13$, limit cycles will exist. The frequency and amplitude can be determined by inspecting the intersections. The prediction based on this method can be verified using time simulation. In this particular case, if the two loci in figure 13.4 intersect, the system is susceptible to limit cycles for any particular value of the rate limit.

13.4.2 Analysis using IQCs

We will attempt to determine the range of $K_{num}$ for which the example system will not limit cycle. This is equivalent to requiring that the $L_2$-gain from $v$ to $w$ in figure 13.3 remains bounded. This test can be performed using the IQC framework as described above. For this purpose, we approximate the rate limiter by the system shown in figure 13.2, with $R = 1$ and $K_{sat} = 1$. The IQCs (13.10), presented in section 13.3, can be applied if we rewrite the system slightly. In particular, a block $\frac{1}{1+s}$ needs to be added. The IQC/β toolbox allows two ways to solve the problem; a script based approach using commands in Matlab, or a graphical approach using block diagrams in Simulink.

Let us first start with the Simulink approach. The toolbox offers a number of blocks that represent relations between signals. For our problem, we need the IQCs for a rate limiter and thus we use the sat_int block, shown in figure 13.5(a). In the dialog box of this element, figure 13.4.2, the IQC relation is defined. The iqcreatelimiter function represents the implementation of the rate limiter approximation IQC, equation 13.10. Besides the sat_int block, another IQC/β-toolbox block called ‘performance’ is included in the Simulink system. This block is used to define over which channel the $L_2$-gain is to be calculated. Its properties are shown in figure 13.4.2. The toolbox will calculate the gain from the output of this block to the input of this block.

Once the Simulink system is defined, the Matlab command iqc_gui with the Simulink filename as argument will calculate the $L_2$-gain for the specified input(s) and output(s).

The second way of computing the $L_2$-gain is by using a script file that directly calls Matlab commands. The IQC/β toolbox creates a new environment called abst especially designed to conveniently implement IQC relations through the use of symbols. This environment is based on the mapping of signals. The explicit relation between the signals of our example system can be determined by cutting out the IQC elements. Figure 13.6 presents the modified system, ready for application of the IQC analysis. For this purpose we have cut the encapsulated rate limiter, as well as the feedback loop.

The following series of Matlab commands perform a stability test on the interconnection. The script makes use of the IQC/β toolbox [MKJR99], the commands should be self
explicatory:

```matlab
>> abst_init_iqc; % initialize abst iqc environment
>> Gp=tf([10],
    [1 2*1.73*0.58 1.73^2]); % define transfer function, Knum=10
>> s=tf([1 0],[1]); % define s
>> w2=sinal; % declare w2 as signal class
>> v=signal; % declare v as signal class
>> y=signal; % declare y as signal class
>> r=v-w2; % define signal r
>> y=iqc_ratemlimiter(r); % define IQC satisfied by encapsulated % rate limiter
>> w=Gp1/(s+1)*y; % define signal w
>> w2=w; % declare w2 and w are identical
>> g=iqc_gain_tbx(v,w) % estimate L2 gain from v to w
```
The stability test results in an infinite $L_2$-gain from $v$ to $w$ for $K_{num} > 11.96$. For lower values of $K_{num}$ the stability test yields a finite $L_2$-gain, indicating global system stability. This is a slightly conservative result, since we know from earlier results that the range of global stability is approximately $K_{num} < 13$. The discrepancy stems from the fact that we have used an approximation of the rate limiter according to figure 13.2. Choosing $K_{sat} = 1$ is a crude approximation, which can be seen when considering the describing functions of a fully saturated rate limiter and its approximation with different values of $K_{sat}$, see figure 13.7(a). For the sake of simplicity, a rate limit of $R = 1.0 \text{ rad/sec}$ is chosen, and an input signal amplitude of $\dot{r} = 1$, which leads to a rate limiter onset frequency of $\omega_{onset} = 1.0 \text{ rad/sec}$. As can be seen in the figure, the approximation with $K_{sat} = 1$ introduces a considerable amount of extra lag and some gain attenuation compared to the exact rate limiter. A representation of this effect in the time-domain is shown in figure 13.7(b). Thus, the system with the approximated rate limiter will reach the ‘neutral stability’ condition 13.12 for lower values of the system gain $K_{num}$.

![Bode Plot](image1)

![Time History](image2)

(a) Describing function representation    (b) Time-domain representation

*Fig.13.7 Comparison of exact rate limiter and approximate rate limiter responses*
14 The use of IQCs for the prediction of PIO

14.1 Problem formulation

In order to use the IQC approach for the prediction of Pilot-Induced Oscillations, the problem needs to be put in a framework suitable for analysis. As a starting point, we rely on some important notions that were generated in earlier attempts to understand the Category II PIO problem. The scenario that is generally seen as critical is described by a pilot-vehicle system, in which the pilot is involved in a tracking task. In the longitudinal case the pilot is involved in pitch attitude tracking, and in the lateral case the pilot is involved in bank angle tracking.

In the Category II PIO case, the question to be answered is: can activation of a rate limiter drive the pilot-vehicle system towards sustained oscillation? Two important assumptions need to be made in order to answer that question; these assumptions are related to pilot behavior and the characteristics of the signals within the pilot-vehicle system. A possible approach is to model the pilot as a pure gain acting on an error signal, without time delay. This very simple model (referred to as synchronous precognitive behavior) was associated with most cases of PIO experienced in flight [KMM95]. In the presence of a sustained oscillation, a pilot will recognize that the error signal (the difference between what he wants—a constant attitude, for example—and what he sees—the actual oscillating attitude—) is sinusoidal and will under certain stressful conditions duplicate this input. When the pilot has adjusted his gain to the linear vehicle dynamics, a mis-adaptation can take place when those dynamics are suddenly changed because of rate limiting onset [Dud97c]. The choice of the pilot gain is important since its value directly influences the stability of the linear closed loop system. Pilot gain depends on a number of factors such as the nature of the task, the aircraft response and the particular pilot characteristics. Pilot gain can be expressed in terms of the phase angle $\Phi_e$ of the transfer function from stick input to aircraft attitude at the crossover frequency.

The second important assumption is related to the characteristics of signals. We are assuming that the tracking command signal has a certain spectral characteristic, and the energy content tends to zero for higher frequencies. Furthermore, we are assuming that during a PIO, the error signals within the pilot-vehicle system are sinusoidal. This seems to be a sensible assumption based on signal analysis of various PIOs.

The problem formulation can now be described in terms of a pilot-vehicle system interconnection structure, see figure 14.1. In this case, the lateral directional control system of the F-18A aircraft is shown, but the example is generic enough to serve as a general example.

Figure 14.1 represents the pilot-vehicle system during a bank angle tracking task. A time-varying pilot model, formulated as an LFT, acts on the error between a commanded bank angle $r_\phi$ and the actual bank angle $\phi$ and generates roll stick force commands $fas$. These
signals feed into a controller $K$, together with the feedback signal picked up by sensors $P_{sens}$ consisting of lateral acceleration, roll rate, yaw rate and sideslip angle. The controller generates command signals to the aileron and the rudder actuators, $P_{act}$. The aileron command signal is rate limited and this operation is depicted as $\Delta_{RL}$. The uncertain airframe dynamics are formulated into an LFT consisting of $P_{AC}$ and $\Delta_U$. The output of the interconnection consists of the bank angle, as well as the error signal $e$ between the actual achieved roll rate and the roll rate that would be achieved in the fully linear case, i.e. without rate limiting. Thus, the block $H_{lin}$ represents the linear mapping from roll stick force to roll rate. The motivation of this channel is that the $L_2$-norm of this signal will be an indication of how sensitive the system is to activation of rate saturation.

There are some important choices to be made in the setup of the analysis. The best combination of parameters for analysis is not known; in fact, the search for this combination is one of the goals of the current research. The time-varying pilot gain LFT can constructed such that it varies between gains corresponding to crossover phase angles of $\Phi_c \in [-120^\circ, -160^\circ]$, for instance. Another important choice involves the reference signal $r_\phi$. It is weighted by a constant $w_r$ that determines the magnitude of this signal. Also, we can incorporate some kind of characterization of the spectral content of this signal. It has been shown in [MKJR99] that such a characterization can be expressed in terms of IQCs as well. In this study, we examine the following possibilities:

**Setup A:** The reference signal is assumed to have dominant harmonics up to a frequency of 5 rad/s.

**Setup B:** The reference signal is assumed to have dominant harmonics up to a frequency of 1 rad/s.

**Setup C:** The reference signal is formed by feeding white noise with energy up to 25 rad/s through a low pass second order filter with a break frequency of 2 rad/s.

An input signal characterization is needed because without it, the signal is assumed to
be in the $L_2[0, \infty)$ space, which is a very general characterization; it can be any signal as long as the energy over a time interval is bounded. In reality, only a smaller sub-class of this space contains signals that can physically occur. Defining a characterization only includes such a sub-class in the analysis and thus makes the region of robust stability less conservative.

The interconnection structure of figure 14.1 can be cast into the generalized description shown in figure 14.2. Here, the fixed and known part of the interconnection is assembled into $G$, and all varying elements described by IQCs are assembled into the $\Delta$-block constituting the lower part of the LFT. With this setup, Theorem 1 can be used to compute L2-gain for any of the channels in the upper part of the LFT. $\Delta$ has the structure $\text{diag}(\Delta_U, \delta_p(t), \Delta_{RL})$ and satisfies the IQC defined by $\Pi_\Delta(j\omega)$:

$$
\Pi_\Delta(j\omega) = \begin{bmatrix}
\Pi_U(j\omega) & 0 & 0 \\
0 & \Pi_{TV}(j\omega) & 0 \\
0 & 0 & \Pi_{RL}(j\omega)
\end{bmatrix}
$$

(14.1)

The actual computation of the L2-gain is performed using the IQC/$\beta$-toolbox for Matlab / Simulink [MKJR99]. This package features several standard IQC-blocks that can either be inserted in a Simulink diagram, or used in a command-mode sequence. The command $\text{iqc\_gain\_tbx}$ estimates the L2-gain for specified inputs and outputs. When the search for IQCs that satisfy eq. (13.3) is infeasible this indicates instability (in the presence of rate limiting elements this may also indicate the existence of limit cycles). We are hypothesizing the following: A failure to pass this stability test is an indication of Category II PIO tendencies.
15 Analysis of existing data

We will apply our analysis to several configurations of the Category II database described in [Dud97a]. The following aircraft models, constituting only lateral dynamics, are subjected to our test:

- YF-16 [Smi79a] This aircraft experienced Category II PIO during its first high-speed taxi run. Following the incident, extensive analysis was performed. A number of key configurations were selected to serve as test cases in a recent study of Category II PIO in [DD97]

- F-18A Extensive data from both in-flight simulation and ground-based simulation is available [Smi79b] and a specific study has been undertaken to examine the effect of rate limiting on handling qualities [DD97].

<table>
<thead>
<tr>
<th>Designation</th>
<th>Aircraft type</th>
<th>Configuration</th>
<th>Rate Limit [deg/s]</th>
<th>Mean PIOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>f1</td>
<td>F-18A</td>
<td>1</td>
<td>none</td>
<td>2.02</td>
</tr>
<tr>
<td>f1-140</td>
<td>F-18A</td>
<td>1</td>
<td>140</td>
<td>3.64</td>
</tr>
<tr>
<td>f1-200</td>
<td>F-18A</td>
<td>1</td>
<td>200</td>
<td>2.38</td>
</tr>
<tr>
<td>f2</td>
<td>F-18A</td>
<td>2</td>
<td>none</td>
<td>2.85</td>
</tr>
<tr>
<td>f2-110</td>
<td>F-18A</td>
<td>2</td>
<td>110</td>
<td>3.46</td>
</tr>
<tr>
<td>y1</td>
<td>YF-16</td>
<td>1</td>
<td>none</td>
<td>3.87</td>
</tr>
<tr>
<td>y1-075</td>
<td>YF-16</td>
<td>1</td>
<td>75</td>
<td>4.41</td>
</tr>
<tr>
<td>y1-090</td>
<td>YF-16</td>
<td>1</td>
<td>90</td>
<td>4.27</td>
</tr>
<tr>
<td>y2</td>
<td>YF-16</td>
<td>2</td>
<td>none</td>
<td>4.21</td>
</tr>
<tr>
<td>y2-090</td>
<td>YF-16</td>
<td>2</td>
<td>90</td>
<td>4.46</td>
</tr>
</tbody>
</table>

Table 15.1 Designation of the configurations that are studied

15.1 Description of the analysis cycle

We will compare results of our IQC-based analysis to data that was gathered during a piloted simulation study performed by DLR and FFA [DD97]. A total of four pilots conducted test runs on both fixed-base and moving-base simulators performing different kinds of bank angle tracking tasks. In table 15.1 the mean PIO ratings from this study are given. They have been assigned to ten configurations: f1, f2 and y1, y2 with different programmed rate limits R positioned in front of the elevator actuators. PIO ratings were given by evaluation pilots directly after each run and the rating system is designed to
indicate the level of PIO susceptibility; lower ratings indicate better handling aircraft and generally a PIO rating greater than 2.0 is considered an indication of PIO tendencies. As is clearly seen for the F-18A cases, a deterioration in PIO ratings is distinctly due to rate limits, since it is the only difference between the different configurations.

For the IQC analysis, some choices are made for a first attempt to reach useful result. First of all, a stability analysis will be performed while omitting the pilot model entirely. Thus, control inputs are generated in an open-loop manner to see whether such commands can destabilize the aircraft. The amplitude of the command signal is chosen as the maximum stick force that can be applied, so the weighting on the stick excitation signal is chosen as $W_{sf} = 75N$.

Secondly, we perform a stability analysis of a combined pilot-vehicle system with the general layout of figure 14.1. Pilot gains for subsequent crossover phase angles are listed in table 15.2. However, as a first approach, we are simplifying the analysis. We are not including model uncertainty ($\Delta U = 0$), we use a fixed pilot gain ($\delta_p$ fixed s.t. $\Phi_c = -160^\circ$), and we disregard the e-channel (see figure 14.2). We compute the $L_2$-gain from the reference bank angle $r_\phi$, weighted with a static gain of $W_c = 10deg$, to the actual bank angle $\phi$.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Pilot gain $K_{pil}$ $\Phi_c = -120^\circ$</th>
<th>$\Phi_c = -160^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>y1</td>
<td>1.23</td>
<td>2.53</td>
</tr>
<tr>
<td>y2</td>
<td>1.11</td>
<td>2.13</td>
</tr>
<tr>
<td>f1</td>
<td>1.83</td>
<td>6.14</td>
</tr>
<tr>
<td>f2</td>
<td>0.86</td>
<td>5.69</td>
</tr>
</tbody>
</table>

*Table 15.2 Pilot gains for selected configurations*

15.2 Results

The Simulink systems used for the $L_2$-gain computation are shown in figures 16.1 (F-18A without pilot model), 16.2 (F-18A with pilot model), 16.3 (YF-16 without pilot model) and 16.4 (YF-16 with pilot model), on pages 107 to 110. In all the cases studied, the implemented IQCs for the rate limiter are positioned before the critical actuator. With critical, we mean the actuator that has the lowest onset frequency (i.e. will be activated first given the pre-defined input characteristics). Furthermore, a block called Performance is included. This block serves as a flag for the toolbox indicating which L2-gain is to be computed. In this block, the IQCs used for input characterization are included according to our setups A, B and C.
The results of the analysis are shown the tables below the figures, as well as the analysis conditions for each of the cases studied (see tables 16.1 to 16.8). Here, $\Phi$, is the crossover angle that defines pilot gain, $W_p$, is the reference signal weighting. The performance blocks refer to the commands used in the setup of figure 16.2.

What we are now interested in, is whether the computed $L_2$-gains shown in the tables can serve as indicators for the assigned PIO ratings. From the results obtained with the setups described above, this analysis method gives us some mixed results. Let us concentrate on the setups in which the pilot models are included, since this should give use the most accurate information. Remember, our hypothesis was that a negative result of the stability test (or: infeasibility of the $L_2$-gain estimation) is an indication of Category II PIO. The results of the $f_1$ configurations in table 16.3 seem to support this hypothesis: Case $f_1$, without rate limiting, which received a mean PIO rating of 2, is accompanied by analysis results that yield finite gains for all three setups. Whereas the cases with rate limiting (140 deg/s and 200 deg/s, respectively) are accompanied by analysis results that yield infeasibility. This result supports our hypothesis as a clear deterioration in PIO ratings was observed for these configurations. The results for the $f_2$ configurations, however, do not support our hypothesis. According to the IQC analysis, the rate limit of 110 deg/s attenuates the maximum $L_2$-gain for setup A and C. For setup B, the $L_2$ gain increases slightly, but stays bounded. This does not correspond with the increase in PIO rating caused by the rate limiter.

Moving to the results of the YF-16 model, table 16.7, we observe that for all $y_1$-cases the IQC analysis yields finite gains, for all setups. However, PIO ratings move from 3.87 to 4.41 and 4.27 because of rate limiting. As rate limiting is included, the $L_2$-gains do increase, but they remain bounded. So again, our hypothesis is not supported. Although it should be noted that the $y_1$ configuration without rate limiting already received a high PIO rating of 3.87, which means that this case suffers from Category I PIO.

Another disconcerting observation can be made in these data. The difference in the computed $L_2$-gains ratio the $y_1$ cases with rate limits of 75 deg/s and 90 deg/s, respectively, do not vary. One would expect this gain to decrease as the rate limit increases. One possible explanation for this would be that the IQCs used to describe the rate limiters are not accurate enough. As we have already seen in section 13.4, the approximation is rather crude. Efforts should be put into improving the quality of the IQCs that are used for the rate limiting elements.

So, checking for robust stability with the simplified setup is not sufficient. Next steps in the analysis would be to incorporate the time-varying pilot model. Also, we will need to include some measure of robust performance. This can be done by including the error-channel in our analysis, as depicted in figure 14.1. Apart from relating PIO susceptibility to infeasibility of the IQC test, we should interpret the size of the $L_2$ gains in some way. This is not straightforward; a large gain from $r_{\phi}$ to $e$ (see figure 14.1) isn’t necessarily an
indication of PIO susceptibility. A nonlinear response that differs substantially from the linear response might still be acceptable. The reference model $H_{\text{lin}}$ really should represent some class of acceptable responses. As long as rate-limited response remains within this class, the configuration is assumed acceptable.
16 Conclusions, Part III

An alternative approach to the analysis of pilot-vehicle systems with the purpose of assessing Category II PIO susceptibility has been proposed. Some preliminary results of an analysis using a simplified setup have been presented. Although the analysis offered interesting insight in the robust stability properties of the system, no definite quantitative relation to PIO tendencies resulted. However, the method has some potential, and future research should be conducted into a better problem formulation to reach better results. A more accurate description of the effect of rate limiting in terms of IQCs is needed. Also, a more explicit way of accounting for the fact that we are dealing with periodic excitation of systems, along the lines of [JKM99], should be investigated.
F-18

Fig.16.1 System setup for IQC analysis of F18 model, direct pilot excitation

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Rate limit</th>
<th>( R ) [deg]</th>
<th>L2-gain ( Setup A )</th>
<th>L2-gain ( Setup B )</th>
<th>L2-gain ( Setup C )</th>
<th>FOSIM PIO Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>f1</td>
<td>/      none</td>
<td></td>
<td>113.3</td>
<td>113.3</td>
<td>31.37</td>
<td>2.02</td>
</tr>
<tr>
<td>f1</td>
<td>/      140</td>
<td></td>
<td>infeas</td>
<td>infeas</td>
<td>infeas</td>
<td>3.64</td>
</tr>
<tr>
<td>f1</td>
<td>/      200</td>
<td></td>
<td>infeas</td>
<td>infeas</td>
<td>infeas</td>
<td>2.38</td>
</tr>
<tr>
<td>f2</td>
<td>/      none</td>
<td></td>
<td>92.53</td>
<td>92.52</td>
<td>32.01</td>
<td>2.85</td>
</tr>
<tr>
<td>f2</td>
<td>/      110</td>
<td></td>
<td>183.7</td>
<td>170.43</td>
<td>66.57</td>
<td>3.46</td>
</tr>
</tbody>
</table>

Table 16.1 Results of the IQC analysis, F-18A, no pilot model

<table>
<thead>
<tr>
<th>( W_{sf} ) [N]</th>
<th>Performance block</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setup A 75</td>
<td>iq_c_domharmonic(1,5[,],9,2)</td>
</tr>
<tr>
<td>Setup B 75</td>
<td>iq_c_domharmonic(1,1[,],9,2)</td>
</tr>
<tr>
<td>Setup C 75</td>
<td>iq_c_white(1,25,1) through low pass filter ( tf(4,[1 \sqrt{3} 4]) )</td>
</tr>
</tbody>
</table>

Table 16.2 Conditions for the IQC analysis, F-18A, no pilot model
**F-18**

![Diagram of F-18 system setup](attachment:image.png)

*Fig. 16.2 System setup for IQC analysis of F18 model, bank angle tracking task*

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Rate Limit</th>
<th>L2-gain</th>
<th>FOSIM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R [deg]</td>
<td>Setup A</td>
<td>Setup B</td>
</tr>
<tr>
<td>f1</td>
<td>/ none</td>
<td>32.26</td>
<td>10.31</td>
</tr>
<tr>
<td>f1</td>
<td>/ 140</td>
<td>infeas</td>
<td>infeas</td>
</tr>
<tr>
<td>f1</td>
<td>/ 200</td>
<td>infeas</td>
<td>infeas</td>
</tr>
<tr>
<td>f2</td>
<td>/ none</td>
<td>28.88</td>
<td>10.58</td>
</tr>
<tr>
<td>f2</td>
<td>/ 110</td>
<td>19.99</td>
<td>12.24</td>
</tr>
</tbody>
</table>

*Table 16.3 Results of the IQC analysis, F-18A, with pilot model*

<table>
<thead>
<tr>
<th>Setup</th>
<th>$\phi_c$ [deg]</th>
<th>$W_c$ [deg]</th>
<th>Performance block</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setup A</td>
<td>-160</td>
<td>10</td>
<td>iqz_domharmonic(1,5,1),9,2</td>
</tr>
<tr>
<td>Setup B</td>
<td>-160</td>
<td>10</td>
<td>iqz_domharmonic(1,1,1),9,2</td>
</tr>
<tr>
<td>Setup C</td>
<td>-160</td>
<td>10</td>
<td>iqz_white(1,25,1) through low pass filter tf(4,[1 $\sqrt{2}$ 4])</td>
</tr>
</tbody>
</table>

*Table 16.4 Conditions for the IQC analysis, F-18A, with pilot model*
YF-16

Fig. 16.3 System setup for IQC analysis of YF16 model, direct pilot excitation

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Rate limit R [deg]</th>
<th>L2-gain Setup A</th>
<th>L2-gain Setup B</th>
<th>L2-gain Setup C</th>
<th>FOSIM PIO Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>y1</td>
<td>/ none</td>
<td>151.59</td>
<td>96.50</td>
<td>56.39</td>
<td>3.87</td>
</tr>
<tr>
<td>y1</td>
<td>/ 75</td>
<td>201.3</td>
<td>98.39</td>
<td>71.89</td>
<td>4.41</td>
</tr>
<tr>
<td>y1</td>
<td>/ 90</td>
<td>202.8</td>
<td>98.25</td>
<td>71.95</td>
<td>4.27</td>
</tr>
<tr>
<td>y2</td>
<td>/ none</td>
<td>162.51</td>
<td>93.89</td>
<td>60.76</td>
<td>4.21</td>
</tr>
<tr>
<td>y2</td>
<td>/ 90</td>
<td>168.55</td>
<td>94.90</td>
<td>62.09</td>
<td>4.46</td>
</tr>
</tbody>
</table>

Table 16.5 Results of the IQC analysis, YF-16, no pilot model

<table>
<thead>
<tr>
<th>Setup</th>
<th>$W_s / N$</th>
<th>Performance block</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setup A</td>
<td>75</td>
<td>iqc_domharmonic(1,5,150,9,2)</td>
</tr>
<tr>
<td>Setup B</td>
<td>75</td>
<td>iqc_domharmonic(1,1,150,9,2)</td>
</tr>
<tr>
<td>Setup C</td>
<td>75</td>
<td>iqc_white(1,25,1) through low pass filter tf(4,[1 √8 4])</td>
</tr>
</tbody>
</table>

Table 16.6 Conditions for the IQC analysis, YF-16, no pilot model
YF-16

Fig.16.4 System setup for IQC analysis of YF16 model, bank angle tracking task

<table>
<thead>
<tr>
<th>Configuration / Rate Limit R [deg]</th>
<th>L2-gain Setup A</th>
<th>L2-gain Setup B</th>
<th>L2-gain Setup C</th>
<th>FOSIM Mean PIO Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>y1 / none</td>
<td>23.09</td>
<td>10.97</td>
<td>8.45</td>
<td>3.87</td>
</tr>
<tr>
<td>y1 / 75</td>
<td>43.27</td>
<td>14.92</td>
<td>16.71</td>
<td>4.41</td>
</tr>
<tr>
<td>y1 / 90</td>
<td>42.64</td>
<td>14.93</td>
<td>16.71</td>
<td>4.27</td>
</tr>
<tr>
<td>y2 / none</td>
<td>13.67</td>
<td>10.79</td>
<td>5.11</td>
<td>4.21</td>
</tr>
<tr>
<td>y2 / 90</td>
<td>infeas</td>
<td>infeas</td>
<td>infeas</td>
<td>4.46</td>
</tr>
</tbody>
</table>

Table 16.7 Results of the IQC analysis, YF-16, with pilot model

<table>
<thead>
<tr>
<th></th>
<th>$\phi_c$ [deg]</th>
<th>$W_c$ [deg]</th>
<th>Performance block</th>
</tr>
</thead>
<tbody>
<tr>
<td>Setup A</td>
<td>-160</td>
<td>10</td>
<td>iqcl3domharmonic(1,5,[],9,2)</td>
</tr>
<tr>
<td>Setup B</td>
<td>-160</td>
<td>10</td>
<td>iqcl3domharmonic(1,1,[],9,2)</td>
</tr>
<tr>
<td>Setup C</td>
<td>-160</td>
<td>10</td>
<td>iqcl_white(1,25,1) through low pass filter tf(4,[1 $\sqrt{8}$ 4])</td>
</tr>
</tbody>
</table>

Table 16.8 Conditions for the IQC analysis, YF-16, with pilot model
PART IV

Analysis using the robust stability and describing function methods
17. Tutorial on methods

In this report we present how two approaches based respectively on robust stability and describing function can be used for the prediction of Category II PIO. Both approaches assume that PIO are characterised by a limit cycle behaviour and can be revealed by a "loss of stability" in some sense.

17.1 Two approaches to Robust Stability analysis of a non linear system

Two methods for PIO analysis based on the robust stability (RS) approach have been investigated. The first method consists of replacing the non linear elements (position and rate saturations) by uncertain linear time-invariant gains and to perform a robust stability analysis of the resulting uncertain linear system. The analysis via time-invariant gains may be, however, optimistic in assessing PIO proneness of a given aircraft configuration. Therefore a further method, based on stability analysis via Lur'e Lyapunov functions is investigated; such method gives a sufficient only condition for asymptotic stability of the original non linear plant, which may lead to conservative results.

17.1.1 Robust Stability analysis of LTI systems subject to uncertain time-invariant parameters.

The main idea of the robust stability (RS) analysis approach is to replace the non linear elements, contained in the model of the closed loop system (essentially rate and position limited elements), by uncertain linear time-invariant gains. The methodology has been applied in the past with a good success to perform sensitivity analysis of flying qualities with respect to uncertainties of physical parameters of the augmented aircraft [CMV92], [CMV90]. The algorithm ROBAN is the software tool developed at CIRA to perform the RS analysis of a LTI system subject to parametric time-invariant uncertainties. ROBAN is based on a polynomial approach, i.e. on the analysis of the characteristic polynomial of the uncertain LTI system.

Let \( n \) be the order of the system and \( p \) the number of uncertain parameters affecting the behaviour of the system itself. Let \( \pi = [\pi_1, \ldots, \pi_p]^T \in \Pi \subset \mathbb{R}^p \) be the vector of uncertain parameters, ranging in the box \( \Pi \), and let \( \pi_0 \in \Pi \) be the vector of nominal values of the uncertain parameters.
Now let \( a(\cdot): \Pi \to R^n, \pi \to a(\pi) \), the vector valued function containing the coefficients of the characteristic polynomial of the system. Denote by \( L: R^n \to P^n, a = (a_1, a_2, \ldots, a_n)^T \to p(s,a) \), where
\[
p(s,a) = s^n + a_1 s^{n-1} + \ldots + a_n, \quad s \in C,
\]
the linear operator mapping a vector of \( R^n \) into \( P^n \), the set of monic polynomials of degree \( n \). Finally define the compound operator \( L_o \) as \( L \circ a \).

The complete behaviour of the uncertain system is described by the following family of monic polynomials
\[
\{ L_o(\Pi) \mid \pi \in \Pi \}.
\]

**Definition 1** Let \( \Pi \subseteq \Pi \); the family of polynomials \( L_o(\Pi) \) is said to be Hurwitz if the roots of \( p(\cdot, a(\pi)) \) are in the open left half of the complex plane for all \( \pi \in \Pi \).

**Definition 2** The stability region \( S \) in the parameter space \( \Pi \) is the set composed of all \( \pi \in \Pi \) such that the roots of \( p(\cdot, a(\pi)) \) are in open left half of the complex plane.

With respect to the above definitions, the following problem is solved by ROBAN.

**Problem 1** Find the stability region \( S \) in the parameter space \( \Pi \).

As we shall see, the idea behind the algorithm ROBAN is that of approximating the stability region \( S \) by the union of boxes in the parameter space \( \Pi \). To check robustness in the given box, it is necessary to have a procedure to solve the following basic problem.

**Problem 2** Given a box \( B \subseteq \Pi \), determine if the family of polynomials \( L_o(B) \) is Hurwitz.

The stability test contained in the algorithm ROBAN allows to solve Problem 2 for any kind of dependence on parameters (the only requirements is that the vector function \( a(\pi) \) has to be continuous in its argument); it implements the ideas proposed in [CMV92], [CM91], [AGGV93]. We shall come back to some technical details of the stability test after the description of the algorithm.

We remark that the algorithm ROBAN computes the boundary of the stability region \( \partial S \) up to a desired resolution. In the sequel we shall use the statement "the box is Hurwitz" to mean that the family of polynomials \( L_o(B) \) is Hurwitz, according to Definition 1.
Procedure 1 (ROBAN)

Set the box $\Pi$ in the List

For each box of the List

If $\text{cond}(\text{box})$ then the box is Hurwitz

Elseif $\|\text{box}\| < \varepsilon$ then box is not Hurwitz

Else divide box in sub-boxes and update List

End

End

End of Procedure 1

After Procedure 1 has terminated, the union of the boxes for which the logical operation \(\text{cond}(\text{box})\) is true is an approximation of the stability region \(S\).

Given a box \(B\), the operation \(\|B\|\) is defined as follows

\[ \|B\| = \max_{i=1,\ldots,2^r} l_i \]

where \(l_i\) is the \(i\)-th side of the box \(B\). In other words the size of the box is given by the length of the longest side of the box.

The logical operation \(\text{cond}(B)\) is true if the sufficient condition for RS of the family of polynomials \(L_a(B)\), implemented via the techniques developed in [CMV92], [CM91], [AGGV93], is satisfied. For the sake of brevity, we do not detail the procedure here, but only recall the fundamental steps. First, one builds a polytope in the coefficients space \(R^n\) which "covers" the image of the coefficient vector over the box \(B\), that is \(P(a(B)) \supseteq a(B)\).

In this case it readily follows that Hurwitzness of the family of polynomials with coefficients in the set \(P(a(B))\) (which is tested via the method proposed in [CM91]) implies Hurwitzness of the original family of polynomials \(L_a(B)\). Moreover, if the way the polytope \(P(a(B))\) is constructed is such that

\[ \text{Lebesgue measure of } P(a(B)) \rightarrow 0 \text{ as } \|B\| \rightarrow 0. \]  

(17.1)

it is guaranteed that the stability region computed via Procedure 1 recovers the exact stability region \(S\) as the procedure parameter \(\varepsilon\) goes to zero. From a practical point of view this means that we can estimate \(S\) up to the desired resolution acting on the parameter \(\varepsilon\). Finally, we remark that the covering procedure implemented in ROBAN satisfies condition (17.1).

The use of ROBAN to detect limit cycles in the non linear pilot-vehicle system under consideration will be presented in section 18.1.1.
17.1.2 Robust Stability analysis using the Popov approach

Since the input-output gains of the non linear elements are actually time-varying, the approach presented in the previous section, based on RS of LTI systems subject to uncertain but time-invariant parameters, may be optimistic in determining PIO proneness of an aircraft. A possible way to validate the previous RS analysis is that of performing extensive time simulations of the non linear system; however this approach may result extremely time consuming. An alternative simple analytical method is presented in this section; this method leads to a sufficient condition for asymptotic stability of the original non linear system. The robust stability analysis is based on the classical Popov approach, which allows to analyse directly the stability properties of a non linear system.

Let us refer to the Lur’e system of Figure 17.1, where the closed loop feedback system has been separated into a non linear part, the diagonal $n$-block $N(\cdot) = \text{diag} (N_1(\cdot), \ldots, N_n(\cdot))$, and a linear part, the matrix $G(s)$. Let us denote by $(A,B,C)$ a state space realisation of the transfer function $G(s)$; we have

$$
\begin{align*}
\dot{x} &= Ax + By \\
u &= -Cx \\
y &= N(u)
\end{align*}
$$

(17.2)

We suppose that each nonlinearity satisfies the sector condition (see Figure 17.2)

$$
L_i u_i^2 \leq u_i N_i(u_i) \leq u_i^2 \quad i = 1, \ldots, n.
$$

(17.3)
The sector condition can be reduced to a standard one by means of the loop transformation

$$\tilde{N}_i(u_i, L_i) = \frac{1}{1 - L_i}(N_i(u_i) - L_i u_i), \quad i = 1, \ldots, n.$$  \hfill (17.4)

Indeed $\tilde{N}_i$ satisfies

$$0 \leq u_i \tilde{N}_i(u_i, L_i) \leq u_i^2, \quad i = 1, \ldots, n.$$  \hfill (17.5)

Substitution of (17.4) into (17.2) leads to the Lur'e system

$$\dot{x} = (A - BCL)x + B(I_n - L)y := \bar{A}(L)x + B(L)y$$
$$u = -Cx$$
$$y = \tilde{N}(u, L)$$  \hfill (17.6)

where $L = diag(L_1, \ldots, L_n)$ and $I_n$ is the $n \times n$ identity matrix. This system is now subject to the standard sector condition.

A sufficient condition for the stability of (17.6)- (17.5) can be found by using the Lur'e type Lyapunov function (see [Vid92])

$$V(x) = x^TPx + 2\sum_{i=1}^{n} \lambda_i \int_{0}^{C_x} N(\sigma)d\sigma,$$

where $P$ is a positive definite matrix and $\lambda_i, i = 1, \ldots, n$ are nonnegative scalars. If there exist $P$ and $\lambda_i$ such that the derivative of $V(\cdot)$ along the trajectories of system (17.6)- (17.5) is negative definite the same system is asymptotically stable. Such
condition can be converted into a Linear Matrix Inequality (LMI) as shown in [Vid92]; this result is summarised in the following theorem (known as Popov Criterion).

**Theorem 1** Popov Theorem - LMI version

System (17.6) - (17.5) is asymptotically stable if there exists a positive definite matrix $P$ and nonnegative diagonal matrices $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, $T = \text{diag}(\tau_1, \ldots, \tau_n)$, such that the following LMI is satisfied

$$
\begin{pmatrix}
\bar{A}(L)P + P\bar{A}(L) & P\bar{B}(L) - \bar{A}(L)C^T \Lambda - C^T T \\
\bar{B}(L)P - \Lambda \bar{A}(L) - TC & -\Lambda \bar{B}(L) - \bar{B}(L)C^T \Lambda - 2T
\end{pmatrix} < 0
$$

**Remark 1** We recall that the result contained in Theorem 1 is a conservative condition for asymptotic stability of the non-linear system (17.6), because it guarantees stability for all non-linearities in the sector (17.5), while the actual non-linear system contains a specific nonlinearity (the one depicted in Figure 17.2) in each feedback loop of Figure 17.1.

If the LMI in Theorem 1 is feasible, then asymptotic stability of the non-linear system in Figure 17.1 is guaranteed. Commercial software is available for solving LMIs problems (see for example the Matlab LMI Control Toolbox).

**Remark 2** Note the difference between the two robust stability analysis. Both methods assume that the non-linear element is approximated by an uncertain input-output characteristic, belonging to a given sector, but the first method adds the further hypothesis that the uncertain characteristic is linear.

17.2 The Describing Function approach

The describing function (DF) method provides an useful tool to predict the possibility of auto-oscillations in a feedback closed loop system [Vid92]. The result is not rigorous, but simple, intuitive and satisfactory in the majority of cases of practical interest.

In its simplest form, the method is applied to systems as in Figure 17.3, that is systems composed of a dynamic linear element, assumed to be sufficiently low-pass, and a purely algebraic non-linear element, characterised by an input-output relationship that is independent of the frequency (the so-called static characteristic).
The presence of a persistent oscillation in the system of Figure 17.3 and that such an oscillation is sinusoidal at the input of the nonlinearity. If we express the input signal as

\[ x(t) = X \sin(\omega t), \]

the output of the nonlinear block will be also a periodic signal, with the same period of \( x(\cdot) \), which can be represented as

\[ y(t) = \sum_{n=1}^{\infty} \left( a_n(X) \cos(n \omega t) + b_n(X) \sin(n \omega t) \right), \]

The Describing Function, \( N(X) \), of the nonlinear element is a complex number, which is a function of the magnitude \( X \) of the sinusoidal signal applied at its input. The modulus of \( N(X) \) is the ratio of the amplitude of the fundamental harmonic of output signal and the one of the input signal, and its argument is the difference in phase angles between the first harmonic of the output signal and the input signal:

\[ N(X) = \frac{1}{X} \left( b_1(X) + ja_1(X) \right) \]

A justification of the hypothesis that the higher order harmonics can be neglected is given by the following practical considerations:

1. Their amplitude is usually less than the one of the fundamental component;
2. The linear part of the system usually behaves like a low pass filter, which tends to reduce the amplitude of higher order harmonics with respect to the fundamental one.

In order to have a sinusoidal limit cycle at the input of the nonlinear block, the following condition must be satisfied.
\[ N(X)G(j\omega) = -1, \]

which is equivalent to the following ones

\[ |N(X)||G(j\omega)| = 1, \quad \phi_1(X) + \arg G(j\omega) = (1 + 2\nu)\pi, \quad \nu \in \mathbb{Z}. \]

When a dynamic non linear element is considered (see the rate saturation case), the analysis becomes complicated. If the non linear system is bounded-input, bounded-output stable, than its response to a sinusoidal input is the composition of two functions [Vid92]:

1. A periodic steady-state component;
2. An aperiodic transient component which decays asymptotically to zero.

We can refer to the periodic persistent part of the output signal and define the DF of the non linear system in the usual way. In general, the DF will be a function not only of the amplitude of the input sinusoidal signal, but also of the frequency of the same input \( x(\cdot) \), that is \( N := N(X, \omega) \). If an analytic expression of this dependence is not available, the DF can be obtained numerically via simulations and implemented as a look up table. The set of equations that defines the conditions of auto-oscillations is in this case

\[ N(X, \omega)G(j\omega) = -1, \]

which can be solved numerically to determine the values of the parameters \( X \) and \( \omega \) characterising the “possible” limit cycles. An \textit{a posteriori} validation of the hypothesis of the DF method must be performed.

In the case of multiple nonlinearities, which cannot be reduced to a unique nonlinearity, we can assume that these elements are independent and compute their DF as if there was no interaction among them (see [And98]). We assume also that the limit cycles are still sinusoidal at the input of each non linear element, all with the same frequency, but generally with different amplitudes and relative phase angles.

Let us consider a closed loop system with \( n \) non linear blocks, rearranged in order to obtain the following scheme, suitable for the analysis
Figure 17.4. Describing Function analysis model for multiple nonlinearities.

where

\[
Xe^{j\alpha} = \begin{bmatrix}
X_1 \\
X_2e^{j\alpha_2} \\
\vdots \\
X_ne^{j\alpha_n}
\end{bmatrix}, \quad N(X, \omega) = \begin{bmatrix}
N_{11}(X_1, \omega) & 0 & 0 & 0 \\
0 & N_{22}(X_2, \omega) & 0 & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & 0 & N_{nn}(X_n, \omega)
\end{bmatrix}
\]

The condition that must be satisfied to guarantee a persistent sinusoidal oscillation in the closed loop system is obtained by equating the loop signals computed by breaking the loop at the input of the multivariable non linear element:

\[
G(j\omega)N(X, \omega)Xe^{j\alpha} = Xe^{j\alpha},
\]

or equivalently

\[
[G(j\omega)N(X, \omega) - I]Xe^{j\alpha} = 0.
\] (17.7)

The solution of this set of equations can be obtained using the currently available software procedures for the determination of the zeros of a complex function. Since the left hand side of (17.7) is strictly non linear, the solution will strongly depend on the initial tentative values of \(X, \omega,\) and \(\alpha.\)

Note that (17.7) is also known as the "harmonic balance equation" and can be extended to equate higher order harmonics.
18. Description of analysis setup

In this section a full description of the steps necessary to perform the analysis according to the presented methods will be given.

18.1 Robust Stability analysis

We have applied the RS techniques (ROBAN algorithm and Popov criterion) to the X-15 case study (see [Mat61]). The original model of the aircraft presents only a rate saturated actuator (fist order model); we have included a stick deflection limit as shown in Figure 18.1 to test the RS theory with a closed loop system with multiple nonlinearities.

Figure 18.1. X-15 closed loop scheme.

The numerical values of the elements in the block diagram are:

\[
\begin{align*}
\theta(s) &= \frac{3.476(s + 0.0292)(s + 0.883)}{s^2 + 0.019s + 0.01}, \\
\delta(s) &= \frac{1}{\tau_r} = 25\text{ sec}^{-1}, \\
H(s) &= \frac{1}{\tau_r} = 25\text{ sec}^{-1}.
\end{align*}
\]

Moreover, the rate limit is 15 deg/sec and \(K_P\) is the pilot gain. The two nonlinearities have been normalised to have unitary slope in their linear operating range (see §17.1.2).

The actual values of the rate limit and position saturation are not considered in the RS analysis because:
1. When we deal with stability analysis versus uncertain time-invariant gains (the ROBAN approach), we only need the knowledge about the characteristic polynomial of the linear closed loop system and about the interval of variation of the parameters.

2. When we deal with absolute stability analysis versus uncertain, time-invariant, memoryless nonlinearities (the Popov approach), we only need the knowledge about the matrix of transfer function of the linear part of nonlinear system and about the sector of variation of each saturations.

18.1.1 The ROBAN algorithm

Let us consider the scheme of analysis for the X-15 aircraft shown in Figure 18.2, where the nonlinear elements have been replaced by linear gains \( L_P \) and \( L_R \), assumed to be uncertain and time-invariant.

![Figure 18.2. Replacement of the non linear elements with the linear gains \( L_P \) and \( L_R \).](image)

It is clear that when the actuator is not rate saturated, then \( L_R = 1 \). If we have an estimate of the maximum input entering the nonlinear element, we can derive an estimate of the minimum value attained by \( L_R \), say \( L_R^{\text{min}} \), so we can bound this uncertain gain in the interval \([L_R^{\text{min}}, 1]\). Similarly, the uncertain gain \( L_P \) representing the stick deflection limit can be bounded in the interval \([L_P^{\text{min}}, 1]\). A third uncertain parameter will be considered in the robust stability analysis, the pilot gain \( K_P \), since this is not known in advance but it can vary depending on several causes. In the following it is assumed that in the real situation to be analysed the pilot gain is constant, i.e. it is held fixed to some particular value during the critical manoeuvre, the actual value maybe depending on the flight phase and the particular pilot himself.

In order to separate the linear part from the two nonlinear blocks, we can rearrange the closed loop nonlinear system of Figure 18.2 according to Figure 18.3.
According to what discussed in section 17.1.1, the first step is to derive the characteristic polynomial of the closed loop linear system of Figure 18.3. In this case

\[ p(s, a) = s^5 + a_1(K_p, L_p, L_R)s^4 + a_2(K_p, L_p, L_R)s^3 + a_3(K_p, L_p, L_R)s^2 + a_4(K_p, L_p, L_R)s + a_5(K_p, L_p, L_R) \]

where

\[ a_1(K_p, L_p, L_R) = 1.72 + \frac{L_R}{\tau_R} \]
\[ a_2(K_p, L_p, L_R) = 5.36 + 1.72 \frac{L_R}{\tau_R} \]
\[ a_3(K_p, L_p, L_R) = 0.22 + 5.36 \frac{L_R}{\tau_R} + 3.48 \frac{(K_pL_p)L_R}{\tau_R} \]
\[ a_4(K_p, L_p, L_R) = 0.053 + 0.22 \frac{L_R}{\tau_R} + 3.17 \frac{(K_pL_p)L_R}{\tau_R} \]
\[ a_5(K_p, L_p, L_R) = 0.053 \frac{L_R}{\tau_R} + 0.090 \frac{(K_pL_p)L_R}{\tau_R} \]

### 18.1.2 The Popov analysis

Let us consider again the system of Figure 18.1 and let us separate the two non linear blocks from the linear part of the system. The two saturation blocks are substituted by two uncertain, time-invariant, memoryless nonlinearities both of them ranging in the sector \([L_i, 1], i = 1, 2\). Firstly, for several combinations of \(L_i\), we carry out the loop transformation as explained in section 17.1.2 to reduce the analysis to nonlinearities varying in the standard sector \([0, 1]\); then, for each fixed value of the pilot gain, we can utilise the sufficient condition introduced by Theorem 1 (an LMI formulation of the
classical Popov theorem) in section 17.1.2, to assure, if satisfied, the absolute stability for the closed loop non linear system. The Matlab commands to use are `ublock`, `udiag` and `popov`. The pilot gain is assumed is constant in the real situation to be analysed, i.e. it is held fixed to some particular value during the manoeuvre, the actual value maybe depending on the flight phase and the particular pilot himself. Since, we expect to obtain conservative results, in the sense that we can have a "safe" estimate of the admissible pilot gains which guarantee absolute stability $\rightarrow$ asymptotic stability $\rightarrow$ absence of limit cycles of the non linear system.

### 18.2 Describing Function analysis

We have applied the DF method to the SAAB and FOSIM databases in [DD97]. With regards to the SAAB database, configuration 410 (with a standard rate limit of 10°/sec) has been considered, with Flight Numbers 537 (Rec. 4), 538 (Rec. 2), 539 (Rec. 2 and 8). Regarding the FOSIM database, we have restricted the analysis to the LATHOS configurations L1 (with rate limit 130 N/sec and 220 N/sec) and L2 (rate limit 110 N/sec), and to the F-18 configuration F1 (with rate limit 80°/sec, 140°/sec, 200°/sec and no rate limit) dealt with in [DD97]. The Simulink models used in the analysis of the two aircraft are shown respectively in Figure 18.4, Figure 18.5 and Figure 18.6.

![SAAB scheme of analysis](image)
The aircraft models in the above databases share a common non linear element, given by the series of a position saturation followed by a rate saturation, in the following indicated as PRS, Position Rate Saturation. It is possible to compute the DF of the series of the two non linear elements by considering the two blocks as a unique non linear system. In this way, the results in terms of limit cycle characteristics detection are much more close to time simulations results than those obtained by using the separate Describing Functions of position saturation and rate saturation. The PRS block is always
dynamic, because of the rate saturation block dynamics. The DF of a dynamic nonlinearity, i.e. one which depends on the input and its derivatives, is also dependent on the input frequency, say $F= F(X, \omega)$. If an analytical expression for this dependence is not available, the DF has to be computed numerically over a finite number of values of $X$ and $\omega$ or, in other words, it can be defined in a table format. The derivation of DF values can be performed off-line with respect to the limit cycles analysis, and depends only on the actual values of position and rate limits. For the PRS both the analytical and numerical approaches have been used for the application of the DF method.

Analytical derivation of the PRS describing function

Let us consider the PRS block in Figure 18.4, and define $R_L$ the value of the rate limit and $P_L$ the value of the input amplitude saturation. According to the definition of sinusoidal input describing function of a nonlinear element, let us consider the output of the PRS block for a sinusoidal input $u(t) = A \sin(\omega t)$ of a given amplitude $A$ and a given frequency $\omega$. The steady state component of this output is a periodic signal of the period of the input sinusoid $u(t)$. The DF of the PRS is the fundamental harmonic of the periodic output signal, which can be obtained in symbolic form from sine and cosine Fourier integrals. In the case of PRS block, the DF is a function of both the input amplitude $A$ and the input frequency $\omega$, say $N(A, \omega)$. The analytical derivation of the DF is facilitated by introducing the following change of variables (see [Pan87] for a similar treatment)

$$\beta = \frac{R_L}{A \omega}; \quad \rho = \frac{P_L}{A} \quad \rightarrow \quad N(A, \omega) = n(\beta, \rho)$$

We can usefully separate the output behaviour according to the different ranges of values of $\beta$ and $\rho$:

- **CASE $\beta \geq 1$ and $\rho \geq 1$**: the amplitude and rate saturations never activate.
- **CASE $\beta \geq 1$ and $\rho < 1$**: the input signal is only amplitude limited and the DF depends only on $\rho$.
- **CASE $\beta < 1$ and $\rho \geq 1$**: the input signal is only rate limited and the DF depends only on $\beta$.
- **CASE $\beta < 1$ and $\rho < 1$**: the input signal is both amplitude and rate limited and the DF depends both on $\beta$ and $\rho$.

The computation of the Fourier integrals is simplified if one separates the possible output shape according to the values of $\beta$ and $\rho$. To this aim some critical values of $\beta$ and $\rho$ are defined below.
\[ \rho_{\text{crit}} = \frac{\pi}{2} \approx 0.844 \]
\[ \sqrt{1 + \frac{\pi^2}{4}} \]
\[ \beta_{\text{crit}}(\rho) = \beta \text{ such that } -\sin(\arccos \beta) + \beta \arccos \beta + \beta \arcsin \rho = \rho \]
\[ \beta_{\ast}(\rho) = \beta \text{ such that } -\sin(\arccos \beta) + \beta \arccos \beta - \beta \arcsin \rho + \beta \pi = \rho \]
\[ \tilde{\beta} = \frac{1}{\sqrt{1 + \frac{\pi^2}{4}}} \approx 0.537 \]

In Figure 18.7 and Figure 18.8 possible output shapes for different cases are presented.

\[ \beta_{\text{crit}}(\rho) \leq \beta < 1 \]
\[ \sqrt{1 - \rho^2} \leq \beta < \beta_{\text{crit}}(\rho) \]
\[ \frac{2\rho}{\pi} \leq \beta < \sqrt{1 - \rho^2} \]
\[ 0 < \beta < \frac{2\rho}{\pi} \]

\[ \rho_{\text{crit}} = 0.844, \rho = 0.5, \beta_{\text{crit}}(\rho) = 0.97, \sqrt{1 - \rho^2} = 0.87, 2\rho/\pi = 0.32 \]

Figure 18.7. PRS output shape for \( \rho \leq \rho_{\text{crit}} \).
Numerical derivation of the PRS describing function

We concentrate on determining \( F(X, \omega) \) for \( N_X \) values of amplitude \( X \), chosen in an interval of input amplitudes \([X_{\text{min}}, X_{\text{max}}]\) that cover the expected operating range of the system, and \( N_\omega \) values of frequency \( \omega \) chosen in an interval of input frequencies \([\omega_{\text{min}}, \omega_{\text{max}}]\) to span the frequency range of interest. In Figure 18.9 the Simulink model used to calculate the DF for a given frequency \( \omega \) and a given \( N_X \)-vector of amplitudes \( X \) is shown. Running this Simulink model inside a loop of \( N_\omega \)-values of \( \omega \) we obtain a \( N_\omega \times N_X \) matrix of complex numbers, which are the DF values corresponding to the assigned \( N_\omega \times N_X \) points. In particular, the output signal \( Y(t) \) of each time-simulation is registered and a spectral analysis is performed. After that the Fourier’s series coeffi-
cients of the first harmonic of the output are derived, the DF is obtained using its definition. Note that, since the rate limiter is a dynamic system, it is necessary to run each simulation for a sufficient time, in order to assure that transients decay to zero and the output of the non linear system is actually periodic.

Figure 18.9. Simulink model for computation of Describing Function.

Referring to the Pilot-Vehicle Systems in Figure 18.4, Figure 18.5 and Figure 18.6, the following set of equations defines the condition of a sinusoidal limit cycle at the input of the pilot gain block

\[ K_p N(X, \omega) G(j\omega) = -1, \quad (18.1) \]

where \( K_p \) is the pilot gain, \( N(X, \omega) \) is the DF of the PRS element, and \( G(j\omega) \) is the transfer function of the remaining linear part of the open-loop system.

An interpolation procedure has been used to find the solution of (18.1), say \( K_p^*, X^* \) and \( \omega^* \), assumed as parameters of the estimated limit cycles.

Time simulations of the Simulink models of Figure 18.4, Figure 18.5 and Figure 18.6 are run for the given values of \( K_p^* \), to make a comparison between the actual (simulated) limit cycle amplitude and frequency and the ones estimated via DF analysis.
19. Results of analysis

In this section we discuss the analysis results provided by the application of:
1. ROBAN algorithm, Popov criterion and Describing Function analysis to X-15 aircraft;
2. Describing Function to SAAB, LATHOS and F-18 databases.

The X-15 test case has been considered to show why, when dealing with systems with multiple nonlinearities in the loop and even in a simple situation, the robust stability analysis approaches may encounter difficulties in assessing a Category II PIO proneness of a given aircraft. The SAAB and LATHOS databases have not been analysed with these approaches due to further analysis complication introduced by the presence of the Simulink rate-limiter blocks (see below).

On the other hand, some very promising results have been obtained using the DF approach to SAAB, LATHOS and F-18 limit cycle detection, and hence to Category II PIO analysis.

19.1 Robust Stability analysis

19.1.1 Results using ROBAN

The algorithm ROBAN approximates, up to a desired resolution \( \varepsilon \), the stability region \( S \) in the parameter space with the union of a finite number of hyper-boxes where the stability condition is satisfied. It is thus possible to reconstruct the boundary \( \partial S \), i.e. the hyper-surface in the parameter space which separates the stability region from the unstable one. The boundary surface \( \partial S \), is the locus in the 3-D parameter space \((L_P, L_R, K_P)\) for which the closed loop system is neutrally stable, i.e. poles with zero real part exist. For a given \( K_P^* \), the neutrally stable frequencies are computed as the imaginary parts of the neutrally stable poles corresponding to the points obtained by the intersection of \( \partial S \) with the plane of equation \( K_P = K_P^* \).

If we simplify the analysis considering only the rate saturated actuator, i.e. \( L_P = 1 \), ROBAN gives the result shown in Figure 19.1, in the parameter plane.
Two regions are visible in the figure, separated by a curve $\partial S$. The region below $\partial S$ identifies the pairs of parameters $(L_R, K_P)$ for which the closed loop system is asymptotically stable, while in the superior region the system is unstable. On the boundary curve $\partial S$ the pairs $(L_R, K_P)$, for which the closed loop system is neutrally stable, i.e. a couple of poles with zero real part exists, are located. For a given $K_P$, the neutrally stable frequencies, computed as the imaginary parts of the neutrally stable poles corresponding to the points $P_S$ and $P_U$ obtained by the intersection of $S$ with the horizontal line of ordinate $K_P$, will be the predicted limit cycle frequency. Note that $P_S$ and $P_U$ are associated respectively to a stable and an unstable limit cycle, and therefore only the limit cycle predicted by the point $P_S$ will be observed in the non linear system. This is confirmed by the result shown in Figure 19.2, where the limit cycle frequency predicted by ROBAN and by a describing function analysis for several values of the pilot gain are compared with the frequency of the limit cycle found in the time simulations. A good agreement of the predicted and actual limit cycles is evident. In particular for this case the results from ROBAN and the describing function analysis can be shown to give the same limit cycle frequency. The line with circles gives the frequency from ROAN and DF, the line with squares gives the frequency from time simulations. From ROBAN and
DF one can also detect the frequency of unstable limit cycles (empty circles), which cannot be found by time simulations.

![Graph](image)

Figure 19.2. X-15 test case. Comparison of limit cycles predicted by ROBAN and describing function and found by time simulations.

Computing the stability boundary for the case when also the stick deflection limit is included is quite easy for the X-15 aircraft. Indeed, if we look at the characteristic polynomial of the closed loop system in Figure 18.3, we note that the uncertain gain $L_P$, associated to the position saturation, always appears multiplied by $K_P$. Since we have already performed robust stability analysis for different values of $K_P$, the stability limit values of $L_P$ are derived from the product $K_P L_P$ for each fixed value of $K_P$. The stability boundary, i.e. the surface $\partial \mathcal{S}$, in the 3-D parameter space is easily obtained by extrapolating the stability boundary curve of Figure 19.1, as shown in Figure 19.3 (note that for clarity of presentation the surface has been "cut" at $K_P=12$). Points above $\partial \mathcal{S}$ identify unstable combinations of the parameters, points below $\partial \mathcal{S}$ identify the stable ones.
It should be noted here that, the intersection of $\partial S$ with the surface defined by the equation $K_P = K_P^*$ is, in the general case, a curve, not a single point. This means that for the linearised system a whole set of couples $(L_P, L_R)$ are found that make the linearised system marginally unstable for the given $K_P^*$.

The problem is here to extract the characteristics of the actual limit cycles developing in the non linear system. In any case, information about the estimated maximum safe value of $K_P$ can be guessed. For the X-15 example, below $K_P \leq 2$ the linearised system is asymptotically stable for any combination of the uncertain parameters. This value is an estimation of a maximum safe $K_P$.

![Figure 19.3. The robust stability boundary of X-15 computed by ROBAN.](image)

The projection of this surface onto the $L_P$ — $L_R$ plane gives the contour level plot shown in Figure 19.4 and Figure 19.5.
Figure 19.4. The robust stability boundary of X-15 computed by ROBAN: a contour level plot.

Figure 19.5. X-15. The contour plot of the stability surface.
For any given pilot gain $K_P$ ROBAN gives a curve in the $L_P$ — $L_R$ plane, along which the linearised system is only marginally stable. In Figure 19.5, each curve divides the $L_P$ — $L_R$ plane in two regions. The region on the right of the curve gives the set of couples $(L_P,L_R)$ where the linearised system is unstable for the given $K_P$, while the region on the left side includes the couples $(L_P,L_R)$ where the linearised system is stable.

As said above, the main problem of robust stability analysis (in a Category II PIO analysis framework) when dealing with multiple uncertain time-invariant real parameters is the difficulty to locate the points on the surface $\partial S$ from which one can estimate the actual limit cycle characteristics. The reason is that in the linear framework the gains $L_P$ and $L_R$ can assume any value on the boundary curve, while in a non-linear framework they are constrained by the condition that they depend on the value of their input. This means that not every combination of couples $(L_P,L_R)$ for a given pilot gain will lead to a limit cycle in the non-linear case. Instead, the dependence of the gain on the input must be taken into account in the analysis. But, at the end, this means that one needs to perform anyway a describing function analysis of the pilot-vehicle system. This drawback limits the applicability of this approach, as an autonomous tool of analysis, even in a simple test case as the X-15 case study.

The Simulink models for SAAB and LATHOS databases show a further complication, due to the presence of a rate limiter instead of a simple rate limited actuator. The rate limiter can be modelled as the limit of the block in Figure 19.6 as $\tau_R$ goes to zero. In other words, the better "equivalent" scheme is that of Figure 19.7 (see [RSG96] and in [BHSB96]).

![Figure 19.6. Block diagram of a rate-limiting symmetric nonlinearity with first order lag.](image-url)
The uncertain time-invariant gain that replaces the relay block in Figure 19.7, say $L_R$, will vary in the interval $[0, +\infty]$. This means that the numerical algorithm ROBAN has to reconstruct the stability region with respect to a set of uncertain parameters in which one of them has an infinite range of variation. Note also that this model is not linearisable, since the characteristic of the non-linear element is discontinuous for zero input. This makes the equivalent gain going to infinity when the input amplitude approaches zero.

19.1.2 Results with the Popov method

If we simplify the analysis considering only the rate saturated actuator, the Popov approach provides a result similar to ROBAN, as shown in Figure 19.8, where the curve labelled as Lur’e is the Popov stability boundary with the Lur’e type Lyapunov function of section 17.1.2. Every couple $(K_P, L_R)$ indicates that for the given $K_P$ the closed-loop system is absolutely stable if the non-linear element has a characteristic bounded in the sector $[L_R, 1]$. For example for $K_P=3$ the pilot-vehicle system is absolutely stable in the sector $[0.3, 1]$.

It should be noted here that the right side of the ROBAN and Popov stability boundary are identical (up to numerical inaccuracies). The left side found with the ROBAN boundary cannot be computed with the Popov approach. Instead, at the left of the point with minimum pilot gain, the Popov boundary is a horizontal line at the value of the minimum pilot gain of the ROBAN boundary. The explanation is the following.

Assume, for example, $K_P=3$. In the sense explained above for the Popov boundary, the point of the left ROBAN boundary with $L_R=0.03$ would imply that the closed loop system is absolutely stable for all non-linear element whose input-output characteristics are...
in the sector $[0.03, 1]$, which is not true, since at least all linear characteristics with gain in $[0.03, 0.3]$ make the closed loop system unstable.

By observing that the stable limit cycles of the X-15 are those located on the left side of the ROBAN boundary, one can conclude that the Popov approach can be used to find an estimate of safe operating regions, but it cannot give information on the actual limit cycles of the closed loop system.

![Figure 19.8. The boundaries of the stability region evaluated via ROBAN (dashed) and Popov (continuous) for the X-15 model.](image)

A similar result can be found from the Popov analysis, when the amplitude saturation is included too, as done with ROBAN. The contour plot of the Popov result is superimposed in Figure 19.9 and Figure 19.10 to the result of ROBAN. Figure 19.9 presents the result for a set of pilot gains, while Figure 19.10 restricts to a particular value of the pilot gain, namely $K_P=6$.

For each fixed value of the pilot gain the Popov boundary is a horizontal line intersecting the ROBAN boundary at the point with $L_P=1$. The Popov result has the following interpretation.

For a given pilot gain, for example $K_P=6$ in Figure 19.10, the Popov method gives the absolute stability sector (box) in the $(L_P,L_R)$ plane of the type $[0,1] \times [L_R,1]$ with lowest $L_R$. It is evident in this case that the Popov method gives a much more conservative result than the ROBAN algorithm.
Figure 19.9. X-15 RS analysis by ROBAN (continuous) and Popov (dashed) methods. Contour plot in the (Lp,Lr) plane.

Figure 19.10. X-15 RS analysis by ROBAN (continuous) and Popov (dashed) methods. Contour plot in the (Lp,Lr) plane. Result for pilot gain Kp = 6.
19.2 Describing Function analysis

The DF technique has been applied to the problem of limit cycles detection for the following databases:

1. SAAB database [Run95], configuration 410, i.e. standard rate limiter, (rate limit 10°/sec), flights 537 (Rec. 4), 538 (Rec. 2), 539 (Rec. 2 and 8);

2. LATHOS database [DD97], configurations L1 (rate limit 130 N/sec and 220 N/sec), and L2 (rate limit 110 N/s);

3. F-18 database [DD97], configuration F1 (rate limit 80°/sec, 140 °/sec, 200 °/sec and no rate limit).

19.2.1 SAAB database

Configuration 410 has been considered, to be compared with configuration 400, a "linear" configuration where only the position limit has been retained.

The results for configuration 410 are shown in the following plots, from Figure 19.11 to Figure 19.22. For each flight considered (#537, #538 and #539), the three plots presented show respectively: 1) the pilot gain $K_P$ versus the frequency of the limit cycle $W_{lc}$ (in rad/sec); 2) the amplitude Amp. (in degrees) versus the frequency of the limit cycle $W_{lc}$; 3) the amplitude Amp. of the limit cycle versus the pilot gain $K_P$. The triangles correspond to the DF analysis results, while the circles to the time-simulations results. Two solid horizontal lines [vertical lines] in the first subplot [in the third subplot] have been drawn in correspondence of the pilot gains for which the phase margin of the linear aircraft model is, starting from the bottom [from the left], 70° or 20°. These have been used in the past as limit pilot gains in other PIO criteria, such as the OLOP criterion developed for Category II PIO analysis (see [DD97]), in order to consider a pilot gain range from low pilot gain, at 70° phase margin, to high pilot gain, at 20° phase margin. It is important to note that when performing a PIO analysis, one should assess both Category I and Category II PIO tendencies. For pilot gains above the one for the lower phase margin the gain margin of the linearised aircraft will be so poor that the aircraft will be certainly subject to Category I PIO. This induces to disregard in the Category II analysis the pilot gains above this value.
Figure 19.11. SAAB Flight 537: pilot gain vs. limit cycle frequency.

Figure 19.12. SAAB Flight 537: limit cycle amplitude vs. limit cycle frequency.
Figure 19.13. SAAB Flight 537: limit cycle amplitude vs. pilot gain.

Figure 19.14. SAAB Flight 538: pilot gain vs. limit cycle frequency.
Figure 19.15. SAAB Flight 538: limit cycle amplitude vs. limit cycle frequency.

Figure 19.16. SAAB Flight 538: limit cycle amplitude vs. pilot gain.
Figure 19.17. SAAB Flight 539_2: pilot gain vs. limit cycle frequency.

Figure 19.18. SAAB Flight 539_2: limit cycle amplitude vs. limit cycle frequency.
Figure 19.19. SAAB Flight 539_2: limit cycle amplitude vs. pilot gain.

Figure 19.20. SAAB Flight 539_8: pilot gain vs. limit cycle frequency.
Figure 19.21. SAAB Flight 539_8: Limit cycle amplitude vs. limit cycle frequency.

Figure 19.22. SAAB Flight 539_8: Limit cycle amplitude vs. pilot gain.
The first comment to the plots is that they look very similar. This descends from the fact that the transfer functions identified from the four sets of flight data, and used in the scheme of Figure 18.4, are very close each other as can be seen in the Nichols plot shown in Figure 19.23.

![Nichols Chart](image)

Figure 19.23: Transfer functions of SAAB flight data 537, 538, 539_2 and 539_8.

In all of the cases there is a good adherence between DF analysis predictions and time-simulations, due to the sufficiently low pass filter characteristics of the linear part of the closed loop systems.

The pilot gain vs. frequency plots show that limit cycles can be generated in the system with medium-high pilot gains, i.e. with pilot gains in the upper part of the range between the low and high pilot gains. The frequency corresponding to the limit cycle arising at the lower pilot gain is about 1.3 rad/s, well within the frequency range of activity of a standard pilot. This is important to note, since in this analysis we have used a pure gain pilot model, i.e. with no cut-off frequency.

The limit cycle amplitude shown in the plots is quite high, in particular for the pilot gain in the above indicated range, the limit cycle amplitude is in the range 30° to 45°, indicating that these limit cycles are potentially disastrous.

In summary the DF analysis predicts that the investigated SAAB configurations have Category II PIO tendency.
This confirms the flight results presented in [Run95], in which PIO ratings are reported. The average PIOR of configuration 410 for the Digital Tracking Task, which is the one more suited to PIO identification, is 3.92, which is a significant degradation from the average PIOR of 2.17 given to the "linear" configuration 400.
19.2.2 LATHOS database

The results for LATHOS database are depicted in the figures from Figure 19.24 to Figure 19.32. For each of the two flight configurations considered (L1 and L2), three plots have been reported: the pilot gain $K_P$ versus the frequency of the limit cycle $W_{lc}$ (in rad/sec); the amplitude $\text{Amp.}$ (in degrees) versus the frequency of the limit cycle $W_{lc}$; the amplitude $\text{Amp.}$ of the limit cycle versus the pilot gain $K_P$. The triangles correspond to the DF analysis results, while the circles to time-simulations results. Two solid horizontal lines [vertical lines] in the first subplot [in the third subplot] have been drawn in correspondence of the pilot gains for which the phase margin of the linear aircraft model is respectively, starting from the bottom [from the left], 70° or 20° (see [DD97]).

![Figure 19.24. LATHOS 1_130: pilot gain vs. limit cycle frequency.](image-url)
Figure 19.25. LATHOS 1_130: limit cycle amplitude vs. limit cycle frequency.

Figure 19.26. LATHOS 1_130: limit cycle amplitude vs. pilot gain.
Figure 19.27. LATHOS 1_220: pilot gain vs. limit cycle frequency.

Figure 19.28. LATHOS 1_220: limit cycle amplitude vs. limit cycle frequency.
Figure 19.29. LATHOS 1_220: limit cycle amplitude vs. pilot gain.

Figure 19.30. LATHOS 2_110: pilot gain vs. limit cycle frequency.
Figure 19.31. LATHOS 2_110: limit cycle amplitude vs. limit cycle frequency.

Figure 19.32. LATHOS 2_110: limit cycle amplitude vs. pilot gain.
In all of the cases there is a good adherence between DF analysis predictions and time-
simulations, due to the sufficiently low pass filter characteristics of the linear part of the
closed loop systems.
In particular from the analysis of Figure 19.24 to Figure 19.26 one can predict that con-
figuration L1 with rate limiter set to 130 °/s can develop limit cycles with medium pilot
gain and at frequencies well within the common frequency range of pilot activity. The
amplitudes of these limit cycles are also high, and all together this indicates a medium
potential of developing dangerous Category II PIO.

Figure 19.27 to Figure 19.29 show that raising the rate limit to 220 °/s the potential for
Category II PIO is reduced, although it is still present. Indeed limit cycles within the
given pilot range are found, but in this case both the minimum pilot gain (i.e. the lower
pilot gain for which a limit cycle is found) and the frequencies of the limit cycles are
higher than in the previous case. The amplitudes of the limit cycles are lower than in the
case with the rate limiter set to 130 °/s, but they are still in a range of high values , 18°
to 26°.
Finally, the analysis of Figure 19.30 to Figure 19.32 let us predict that this configuration
should be free from Category II PIO potential, since the pilot gains for which limit cy-
cles develop are above the range indicated by limit the phase margins.

It should be noted that these predictions confirm the predictions given in [DD97] by the
OLOP criterion, which is considered a validated Category II PIO criterion. This is a
satisfactory indication that the proposed approach can be useful in Category II PIO
analysis.

19.2.3 F-18 database
The results of the DF analysis for configuration F1 of the F-18 aircraft are shown in
Figure 19.33, Figure 19.34 and Figure 19.35. The DF analysis results for different val-
ues of the rate limiter element of Figure 18.6 are reported with different markers (open
symbols). The parameters of the limit cycles found by time simulations are also re-
ported for comparison (filled markers). A good agreement is found again. In Figure
19.33 several horizontal lines at different values of the pilot gain have been drawn, each
one corresponding to a particular value of the phase crossover of the linearised system
in the range [-160° , -110°]. From the figures the effects of raising the rate limit can be
expressed as:
1. the lower pilot gain for which a limit cycle is possible is increased
2. the frequency range of the predicted limit cycles within the phase crossover range
   [-160° , -110°] is reduced (higher minimum frequency and lower maximum fre-

ency). Note that even when the the rate limit is set to 200°/s, the restricted fre-
quency range found, [1.6, 5.4] rad/s, is within the frequency range usually associated to PIO events.

3. limit cycles of increasing amplitude are found for the same frequency
4. limit cycles of decreasing amplitude are found for the same pilot gain

Overall the DF analysis of configuration F1 shows a strong cat.II PIO potential with the rate limit equal to 80°/s, a medium cat.II PIO potential with the rate limit equal to 140°/s and 200°/s, and no cat.II PIO potential when the rate limit is absent.

Figure 19.33. F-18 F1: pilot gain vs. limit cycle frequency.
Figure 19.34. F-18 F1: limit cycle amplitude vs. limit cycle frequency.

Figure 19.35. F-18 F1: pilot gain vs. limit cycle amplitude.
20. Conclusions, Part IV

We have presented two approaches for the analysis of Category II PIO, both based on the assumption that PIO are identified by a loss of stability, and can be characterised as limit cycles of the pilot-vehicle system.

The first approach, based on two methods for Robust Stability analysis, has been shown to give good results only for the simplest pilot-vehicle system configurations analysed. In particular, adding an amplitude limitation in front of the rate limit makes the solution by this approach unfeasible.

The second approach, based on the Describing Function method for limit cycles search, has been shown to give good results even with the more complex pilot-vehicle system analysed. Therefore it is considered that this approach can be fruitfully used in the prediction of Category II PIO.

The method presented here gives indications on the potential to develop Category II PIO of a given configuration by looking at the limit cycles that can develop in the close-loop system. In particular a non-linear element, the PRS (series of Position and Rate Saturation), has been considered, which is common to the SAAB, LATHOS and F-18 databases in [Run95] and [DD97].

The method requires, first to define a pilot range producing high (low pilot gain) to low (high pilot gain) phase margins of the linearised system, and then to perform a limit cycle search via the Describing Function approach. If limit cycles are found for pilot gains within the above range, then a potential for Category II PIO is predicted. This potential can be low to high according to several parameters, e.g. the value of the lower pilot gain for which a limit cycle is found, as compared to the indicated analysis range of pilot gains, the frequency range and the amplitudes of the predicted limit cycles.

As shown by the examples presented, the results obtained with the method are satisfactory.

Finally, it should be said that the analysis method based on the Describing Function is not limited to the PRS type of non-linear element but it can work equally well for other kind of non-linear elements.
21 General Conclusions

In this report new analysis methods for the prediction of Category II PIOs have been described and applied to existing PIO data bases. The analysis methods that have been assessed are:

- Bifurcation analysis
- Analysis using classical and extended mu tools
- IQC analysis
- Robust stability analysis

For each method, a short tutorial has been presented first, a description of the analysis set-up is then given, and the results of the application to one or to both SAAB and FOSIM data bases are provided.

Concerning the bifurcation approach, the envelopes and amplitudes of the stable limit cycles have been computed as functions of the pilot gain. In most cases, the limit cycles have been found without numerical problems by the ASDOBI software package. For some cases, the asymptotic solution is more complex (period doubling bifurcation,...) and a failure of the limit cycle search by ASDOBI has been encountered. For those cases, a simulation has been performed and the "envelope" of the trajectory has been plotted. The preliminary application to the F-18 aircraft of the FOSIM data base shows a large jump in limit cycle amplitude as the pilot gain is increased, indicating that a significant change in the PVS structural stability. The results also provide the combinations of actuator rate limit, command frequency and amplitude leading to this stability change. Although a correspondence can be made with the flying qualities cliffs which characterize sudden large changes in aircraft motions associated with slight changes in pilot activity, further analysis work will be necessary to evaluate the potential of the approach and to correlate the results with the OLOP criterion.

The approach using classical and extended mu tools, with the accompanying LFT formalism, proved to provide an interesting framework for PIO analysis. The methods performs well in the presence of simultaneous amplitude and rate saturation. Checked on the F-18 lateral system, it gave good results -rather similar to OLOP- when the rate saturation is positioned before the amplitude saturation. In the case where these elements are switched, some problems occur in the first harmonic approximation used for the saturation modeling. Solving this needs some additional effort. Some ideas have been presented on using the mu-upperbound as well. However, preliminary results appear to be quite conservative.
In part III, the application of integral quadratic constraints to pilot-vehicle systems with rate limiters has been examined. The use of IQCs allows for a formal inclusion of this type of nonlinear behavior in the stability analysis. It is also possible to extend the method to robust performance tests using weighting of characteristic signals. The challenge of this method lies in finding a good problem formulation such that computed $L_2$-gains of weighted signals can be correlated to PIO tendencies. With the formulation presented in this report this correlation is not successful. However, some understanding of how the problem formulation may be improved in order to reach better results has been developed. The inclusion of time-varying pilot gain, and a performance channel based on comparison between the rate-limited vehicle and a pre-defined set of acceptable vehicles are proposed. Also, a search for more accurate IQCs that describe the effects of rate limiting may benefit the method.

Of the two approaches presented in part IV, the second approach based on the describing function method for limit cycles search has proven to be good results even with the more complex pilot-vehicle systems. The methods gives indications on the potential to develop Category II PIO of a given configuration by looking at the limit cycles that can develop in the closed-loop system. In particular a nonlinear element, the PRS (series of position and rate saturation) has been considered, which is common to the SAAB, LATHOS and F-18 databases. As shown by the examples presented in part IV, the results obtained with the method are satisfactory for the range of configurations studied.

21.1 Practical considerations

The work presented in this report has a rather theoretical nature. For practical use of any of the insights gained from these theories a considerable number of additional steps are required. It is desirable that for engineering purposes, these insights are translated into clear and concise parameter mappings or plots. Moreover, relating such mappings to predicted PIO ratings or handling qualities ratings (such as Cooper-Harper ratings, or Level 1 through Level 3 Handling Qualities) is desirable. In such a setting, the insights shall be easy to use as design guidance criteria, test and evaluation criteria and possible clearance/acceptance criteria. It is noted, however, that the work presented in this report serves as a first step in such a process of criteria development. It forms the theoretical fundament supporting the future development of a sophisticated set of criteria.

With this in mind, the work of the GARTEUR action group FM/AG12 will continue and is directed towards these goals. The next step will be the application of the four analysis methods, together with the OLOP criterion, to the ADMIRE model. ADMIRE represents a realistic, full combat aircraft, developed by FFA. During this exercise a large effort will be made to improve the presentation of the results provided by each method in
order support a thorough evaluation and comparison. The relevance of any new analysis approach depends indeed on its ability to be correlated with existing data and more mature criteria (time-domain Neal-Smith, OLOP), as well as with new experimental data.
References


