GARTEUR Open

Optimization problems, methods, and techniques in flight control design: an inventory for Task 1 of GARTEUR FM(AG10)

by

FM(AG10)

GARTEUR aims at stimulating and co-ordinating co-operation between Research Establishments and Industry in the areas of Aerodynamics, Flight Mechanics, Helicopters, Structures & Materials and Propulsion Technology
GARTEUR Open

Optimization problems, methods, and techniques in flight control design: an inventory for Task 1 of GARTEUR FM(AG10)

by

FM(AG10)

This report has been prepared under auspices of the Responsables for Flight Mechanics, Systems and Integration of the Group for Aeronautical Research and Technology in EUROpe (GARTEUR)
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Summary

GARTEUR Flight Mechanics, Systems and Integration Action Group FM(AG10) "Multi-variable Optimization Techniques for Experimental and Conceptual Design" is developing an interface between optimization problems on the one side and optimization methods and techniques on the other side. Optimization problems in FM(AG10) are taken from two domains in flight mechanics: flight control design and experiment design for flight mechanics. To allow a first design of the interface, an inventory on optimization problems, methods, and techniques in flight control design and in experimental design has been carried out, see the accompanying report [5]. The present report presents an inventory of optimization problems, methods, and techniques in flight control design.
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FM(AG10) principal persons

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<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$t_0$</td>
<td>starting time</td>
</tr>
<tr>
<td>$t_f$</td>
<td>final time</td>
</tr>
<tr>
<td>$t_k$</td>
<td>k-th time tag</td>
</tr>
<tr>
<td>$w_{f_1}, w_{f_2}$</td>
<td>disturbance vectors</td>
</tr>
<tr>
<td>$\hat{x}$</td>
<td>estimate or prediction of the variable $x$</td>
</tr>
<tr>
<td>$E$</td>
<td>expectation</td>
</tr>
<tr>
<td>$\tilde{E}, E_{\text{max}}$</td>
<td>evaluation functions</td>
</tr>
<tr>
<td>$G_x$</td>
<td>gravity force in $F_B$ x-direction</td>
</tr>
<tr>
<td>$G_y$</td>
<td>gravity force in $F_B$ y-direction</td>
</tr>
<tr>
<td>$G_z$</td>
<td>gravity force in $F_B$ z-direction</td>
</tr>
<tr>
<td>$H$</td>
<td>Hamiltonian</td>
</tr>
<tr>
<td>$I$</td>
<td>identity matrix of appropriate order</td>
</tr>
<tr>
<td>$J$</td>
<td>cost function or cost functional</td>
</tr>
<tr>
<td>$Q_i$</td>
<td>weighting matrix in cost function, $i = 1, 2, 3$</td>
</tr>
<tr>
<td>$R$</td>
<td>weighting matrix in cost function</td>
</tr>
<tr>
<td>$T$</td>
<td>transpose (superscript)</td>
</tr>
<tr>
<td>$T_c$</td>
<td>engine thrust along z body axes</td>
</tr>
<tr>
<td>$W_E$</td>
<td>commanded engine thrust along z body axes</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Dirac delta function, Dirac delta distribution</td>
</tr>
<tr>
<td>$\delta_{E,c}$</td>
<td>elevator deflection command</td>
</tr>
<tr>
<td>$\delta_F$</td>
<td>flaperon deflection</td>
</tr>
<tr>
<td>$\delta_{F,c}$</td>
<td>flaperon deflection command</td>
</tr>
<tr>
<td>$\delta_H$</td>
<td>symmetric stabilator</td>
</tr>
<tr>
<td>$\omega$</td>
<td>inertial rotational velocity</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>time lag</td>
</tr>
<tr>
<td>$\rho_i$</td>
<td>relative degree of the $i$-th regulated output</td>
</tr>
<tr>
<td>$\Sigma_F$</td>
<td>sum of all externally applied forces</td>
</tr>
<tr>
<td>$\Sigma_M$</td>
<td>sum of all applied torques</td>
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1 Introduction

In the next decade, with the expansion of concurrent engineering, there will be a growing need for exchange of data and information in projects in which several industries and research institutes participate. An example is the multidisciplinary design of aero-elastic aircraft. In such a project, each partner contributes to the overall design, with the envisaged goal of making a better aircraft. The diversity and complexity of the envisaged system leads to multi-objective optimization for complex multi-domain applications. In many fields of aerospace engineering, optimization algorithms have already been developed. These algorithms are usually tailored for specific applications. Each organization, in the industry and among research institutes, has its own tools and strategies with respect to optimization. For a successful long term cooperation, these optimization tools and strategies will have to be (made) compatible, in order to achieve multi-objective optimization such that the design objectives are satisfied. In the field of multidisciplinary design of aircraft this encompasses competitive optimization of parameters and performance indicators.

One of the long term goals of cooperation in GARTEUR is to design and implement universal multi-objective multi-domain optimization software integrating existing algorithms. A first step towards this long term goal is made by facilitating the exchange of data and information between optimization algorithms and different applications.

GARTEUR Flight Mechanics, Systems and Integration Action Group FM(AG10) ”Multivariable Optimization Techniques for Experimental and Conceptual Design” is developing an interface between optimization problems on the one side and optimization methods and techniques on the other side. The interface is an intermediate layer between mathematically formulated problems from specific applications and multi-purpose numerical optimization methods and techniques. The interface will support the handling of competitive indicators. It will be ready for extension to an intelligent interface, which could advise the user in choosing between optimization methods and techniques and in handling competitive indicators. This extension will not be pursued by FM(AG10).

The objective of FM(AG10) is to define a convenient interface standard between applications and algorithms. The interface standard includes standardised data structures. The validated design of the interface will be such that it can be implemented easily in various numerical environments. In addition, the interface is designed for easy incorporation into computer integrated design and research environments.

Optimization problems in FM(AG10) are taken from two domains in flight mechanics: flight control design and experiment design for flight mechanics. This report deals with optimization problems, methods, and techniques in flight control design. Optimization problems, methods, and techniques in experiment design are treated in the accompanying
report [5]. Further information about FM(AG10) can be found in the project document of GARTEUR FM(AG10) [4].

Optimization methods and techniques are one of the cornerstones of the active research area of the design of automatic flight control systems. The optimization methods and techniques appear in two ways. Many flight control design problems can be seen as optimization problems. The running GARTEUR Flight Mechanics, Systems and Integration Action Group FM(AG08) "Robust Flight Control in a Computational Aircraft Control Engineering Environment" has defined the flight control design challenges RCAM in [1] and HIRM in [2]. Twenty two design teams from industry, research laboratories, and universities in several European countries have worked on solutions to these design challenges. The design challenges can be seen as optimization problems with multiple objectives. Optimization problems, methods, and techniques also frequently appear in modern flight control design methods. A few examples are $H_{\infty}$-analysis/synthesis, gradient algorithms, and the Kalman filter.

This report presents an inventory of optimization problems, methods, and techniques in flight control design. The inventory is a part of the work defined by GARTEUR FM(AG10). In the accompanying report [5] a complementary inventory on optimization problems, methods, and techniques can be found, which is for a large part aimed at experiment design. The present inventory contributes to the development of the interface in two ways:

- The optimization problems give directives to the type of problems which should be solvable using the interface. The optimization methods and techniques give directives to the type of methods and techniques which should be incorporated in the interface.

- The inventory of optimization problems will be used later in the project to define a benchmark optimization problem. This benchmark will be used to validate the design of the interface.

The papers for the inventory on flight control design have been selected from the literature on flight control design. The selection criterion has been whether or not in a paper optimization methods and techniques are applied. This report is not an attempt to give an overview of general mathematical optimization problems nor an attempt to give an overview of general mathematical optimization methods and techniques. For an initial overview of general mathematical optimization problems, methods and techniques one may consult for instance Chapter E04 of the introductory guide for the NAG Fortran Library [8].

The modelling equations of the aircraft are the main constraints to be satisfied by a
solution of the optimization problem. Therefore the inventory is centered around the following three topics:

- models of aircraft,
- optimization problems,
- optimization methods and techniques.

The report is structured in three parts. Part I is the core of the report. There, in Chapter 2, the structure is described which is used to present the results found in literature. Furthermore, Part I contains the results from the literature study on models of aircraft (Chapter 3), optimization problems (Chapter 4), and optimization problems and techniques (Chapter 5). Part II contains a description of some of the papers in literature. In each chapter a paper is reviewed along the structure developed in Chapter 2. Part III contains conclusions and recommendations.

The notation and symbols in the papers are adapted to our notation and symbols for a uniform overview. Our notation and symbols are the European-wide supported ones in the Communication Handbook of GARTEUR FM(AG08) [3]. Only when the Communication Handbook does not provide notation or symbols, additional notation or symbols are introduced in the present report. The additional notation and symbols can be found in the "List of symbols".
Part I

Structure of and results from the literature study
2 Structure for the description of literature on optimization problems, methods, and techniques

2.1 Introduction

The inventory of optimization problems, methods, and techniques is presented in two parts. Some papers are reviewed in Part II of the report. An overview of all the literature that has been studied is given in Part I. Both the review and the overview are presented in a structured way. In each chapter of Part II a paper is reviewed along the following main lines, which form the sections of the chapter in which the paper is reviewed:

1. Model of the aircraft
2. Mathematical structure of the model
3. Optimization problem
4. Mathematical formulation of the optimization problem
5. Optimization method or technique
6. Remarks
7. Further references
8. Comments

When a topic is not treated in a specific paper, this will be mentioned explicitly. The topics above are briefly described in the remaining sections of this chapter. The structure that is developed in the present chapter is also used to give an overview of all literature that has been studied.

2.2 Model of the aircraft

In the section "Model of the aircraft" a mathematical model of the aircraft is presented. The section consists of two subsections: "Modelling assumptions" and "Modelling equations". In the subsection "Modelling assumptions" the basic properties of the aircraft are given and the assumptions on the aircraft are described. The modelling equations of the aircraft are described in the subsection "Modelling equations". They usually consist of several parts:

- equations of motion,
- aerodynamic equations,
• actuator equations,
• wind equations,
• sensor equations.

These parts are described explicitly. Parts that are not specified in the paper, are men-
tioned explicitly.

2.3 Mathematical structure of the model

The optimization method or technique can be applied to other, more general models than
the specific aircraft model in the paper only. In many papers the model of the aircraft is
presented in a general form. In the section "Mathematical structure of the model" this
general form is taken from the paper under consideration.
The description of the mathematical structure of the model includes the identification of

• the state vector for the model of the aircraft,
• the control inputs,
• the exogeneous inputs,
• the measurement outputs,
• the regulated outputs, and
• general structure of the model.

2.4 Optimization problem

In the section "Optimization problem" the optimization problem is discussed in its most
general form. Three topics are addressed:

• the data for the optimization problem,
• further assumptions needed to state the optimization problem, and
• the statement of the optimization problem in terms of the performance of the aircraft.

2.5 Mathematical formulation of the optimization problem

The optimality conditions in the optimization problem of the previous section are usually
not precisely defined. Hence, in literature, another optimization problem is solved. We
shall refer to the latter optimization problem as the approximate optimization problem,
motivated by the following two observations:
• a solution of the approximate optimization problem may be interpreted as a solution of the original optimization problem, and
• there exist optimization methods and techniques to approximate a solution of the approximate optimization problem in a numerically stable way.

For an original optimization problem, one can formulate various approximate optimization problems. An important part of the review in the section "Mathematical formulation of the optimization problem" is to comment on the way in which a solution of the approximate optimization problem may be interpreted as a solution of the original optimization problem.

The formulation of the approximate optimization problem involves the choice of

• a criterion function to be optimized, and
• constraints to be satisfied.

In the section "Mathematical formulation of the optimization problem" we present the criterion function and constraints that are given in the reviewed paper.

Criterion functions and constraints usually involve design parameters. The design parameters that appear in a reviewed paper, will be listed explicitly.

2.6 Optimization method or technique

In the section "Optimization method or technique" we review the optimization method or technique which is used in the paper to solve the approximate optimization problem. Short algorithms are stated; for longer algorithms usually an outline is given. Some papers present a closed formula for the solution of the approximate optimization problem. In this case the closed formula is given or a reference to it is made when the formula would take a lot of space.

The following topics get special attention.

• Additional design parameters may be involved in the method or technique. They are listed explicitly.
• The choice of the various design parameters is discussed.

2.7 Remarks

In the section "Remarks" all other parts of the paper that are relevant for our inventory, are summarised. For example, remarks about simulation may be included. Also evaluation functions may be presented, which are used to evaluate the performance of the solution of the optimization problem with respect to objectives different from those stated in the optimization problem.
2.8 Further references

The section "Further references" contains references besides the main reference. The relevance of the references is explained.

2.9 Comments

In the section "Comments" we give comments on the reviewed model of the aircraft, the reviewed optimization problem, or the reviewed optimization method or technique.
3 Results on models of aircraft in flight control design

3.1 Introduction

Various mathematical models of aircraft are used in flight control design. The models have often been derived from physical laws, making assumptions on the aircraft and its environment. From a mathematical point of view, many aircraft models have the same general form.

In this chapter the modelling assumptions, the modelling equations, and general forms of the aircraft models that have been found in the literature, are presented. Section 3.2 contains an overview of the modelling assumptions, Section 3.3 of the modelling equations, Section 3.4 of the general forms of aircraft models.

3.2 Modelling assumptions

Modelling assumptions which have been encountered in the literature are:

1. the aircraft is a rigid body,
2. earth is fixed in space,
3. earth is flat,
4. the mass and mass distribution of the aircraft are constant over time,
5. the inertia of the aircraft is constant over time,
6. the $X_B, Z_B$-plane is a plane of symmetry,
7. the inertia cross-products $I_{zz}$ and $I_{zy}$ are equal to zero,
8. gravity force is constant
9. flight takes place in a vertical plane over a flat earth,
10. the aerodynamic coefficients have simple modelling equations,
11. the density of air is constant, independent of the height,
12. the actuators have simple models,
13. the thrust force is parallel to the $X_B$-axis,
14. the thrust force is along the $X_B$-axis,
15. wind is absent,
16. wind model is deterministic,
17. wind behaves as a fixed stochastic process,
18. the sensors are perfect,
19. the noises of the various sensors are independent,
20. the noise of a sensor is independent of time,
21. the noise of a sensor behaves as a Gaussian white noise process,
22. the aircraft can be modelled by a linear system of equations.

Assumptions 10, 16, 17, 21, and 22 will be specified in the next section, when the modelling equations are described. Notice that assumption 7 is satisfied whenever assumption 6 is fulfilled. Standard assumptions for all models are assumptions 1-8.

The list of modelling assumptions above can be extended with modelling assumptions that have not been encountered in the literature studied. For example, the gyroscopic effect due to engines is often neglected in the mathematical model. However, when the aircraft is yawing, the gyroscopic effect can be significant. Indeed, in this case the rotary parts of the engine induce a torque in pitch, even if the thrust is perfectly aligned on the x-axis. It can be expressed as $I_x \text{ eng} \tau_{\text{eng}}$, where $I_x \text{ eng}$ is the moment of inertia of the rotary parts, $\tau_{\text{eng}}$ is the rotation velocity of these rotary parts and $\tau$ is the yaw velocity of the aircraft. The rotation velocity of the engine $\tau_{\text{eng}}$ is quite larger than $p$, $q$, and $r$, so if $I_x \text{ eng}$ is not too small, the gyroscopic effect cannot be neglected.

3.3 Modelling equations

The modelling equations for aircraft in the literature consist of five parts:

- equations of motion,
- aerodynamic equations,
- actuator equations,
- wind equations, and
- sensor equations.

3.3.1 Equations of motion

Mathematical modelling of the motion of the aircraft leads to a nonlinear system of equations. Besides nonlinear systems, linear systems appear as well in the literature. First nonlinear equations of motion will be presented, next linear equations of motion will be discussed.
Derivations of the equations of motion, which use arguments from physics, can be found in several books and papers, e.g., McRuer-Ashkenas-Graham [40] and Wise-Godhwani-Gu-Dierks-Kerckemeyer-Tang [63]. Taking the standard assumptions 1-8, it follows from Newton's second law that

\[
m\dot{V}_B = -\omega \times mV_B + \Sigma_F, \quad \dot{\omega} = I^{-1}(\omega \times I\omega) + I^{-1}\Sigma_M,
\]  

(3.1)

where \(\omega\) is the inertial rotational velocity, \(\Sigma_F\) the sum of all externally applied forces, and \(\Sigma_M\) the sum of all applied torques about the centre of gravity. The externally applied forces are the gravitational forces, thrust forces, and aerodynamic forces. The externally applied torques are the aerodynamic torques and the engine torques. Assume that assumption 14 is also satisfied. Substituting \(V_B = (u, v, w)^T\) and \(\omega = (p, q, r)^T\) into (3.1), we obtain the nonlinear system of equations (see Wise-Godhwani-Gu-Dierks-Kerckemeyer-Tang [63]):

\[
\begin{align*}
\dot{u} &= ru - qw + G_x + \frac{1}{m} F_{xA} + \frac{1}{m} T, \\
\dot{v} &= pw - ru + G_y + \frac{1}{m} F_{yA}, \\
\dot{w} &= qu - pv + G_z + \frac{1}{m} F_{zA}, \\
\dot{p} &= -L_{qr}qr - L_{pq}pq + \frac{L_A L_x + N_A L_z}{I_x I_y} - \frac{L_A L_z - N_A I_x}{I_y I_z}, \\
\dot{q} &= -M_{pr}pr - M_x r^2 + M_y, \\
\dot{r} &= \frac{-L_q q r - N_{pq}pq + N_A L_x + L_A L_x}{I_x I_y} - \frac{L_A L_x - N_A L_y}{I_y I_z},
\end{align*}
\]  

(3.2)

where the coefficients \(L_{qr}, L_{pq}, M_{pr}, M_x, r^2, N_{qr},\) and \(N_{pq}\) are constants depending on \(I_x, I_y, I_z,\) and \(I_2\); \((G_x, G_y, G_z)^T\) is the gravity force in body axis coordinates, and \(T\) denotes the engine thrust force. Expansion of the gravity force \((G_x, G_y, G_z)^T\) leads to the addition of three differential equations for \(\phi, \theta,\) and \(\psi\) (see McRuer-Ashkenas-Graham [40]).

In the RCAM problem [1] and the HIRM problem [2] the modelling assumptions 1-5, 7, 8, and 17 are used to derive the equations of motion. This leads to the addition of engine moments in the modelling equations (3.2). The equations of motion in Singh-Steinberg-DiGirolamo [55] have been derived from (3.2) by making further assumptions (see Hacker-Oprisiu [24] for the derivation and the assumptions).

A constraint to the equations of motion appears in Lu-Pierson [35], in which the angle of attack \(\alpha\) is required to be between specified bounds.

In many papers it is assumed that the aircraft flies in a vertical plane over a flat earth (assumption 9). This assumption leads to a considerable reduction of the system (3.2). Equations of motion for flight in a vertical plane over a flat earth appear in Mulgund-Stengel [42], Khan-Lu [28], and Lu-Pierson [35], for example. The equations of motion are of the form as in Part II, equation (7.1) (see below).
Further reduction of the equations of motion may be obtained by considering energy-state models for flight in a vertical plane. Energy-state models contain less variables than the models in the previous paragraph. Examples of energy-state models can be found in the papers Wu-Guo [64], Rao-Mease [49], Subbaram Naidu-Calise [58], Kumar-Seywald [33] (Mayer form), Erzberger-Lee [17], and Barman-Erzberger [10].

Linear equations of motion can be obtained from (local) linearisation of the nonlinear equations of motion. The linearisation procedure is often not described in the literature. Examples of books and papers in which the linearisation procedure is described, are McRuer-Ashkenas-Graham [40], Hacker-Oprisiu [24], and Wise-Godhwani-Gu-Dierks-Kerkemeyer-Tang [63]. In a classical linearisation procedure two linear models of the aircraft are made: one for the longitudinal behaviour of the aircraft and one for the lateral behaviour of the aircraft.


3.3.2 Aerodynamic equations

In the RCAM problem [1] the aerodynamic forces and aerodynamic moments are

\[
\begin{align*}
F_x &= \frac{1}{2} \rho V_A^2 S C_x, \\
F_y &= \frac{1}{2} \rho V_A^2 S C_y, \\
F_z &= \frac{1}{2} \rho V_A^2 S C_z, \\
L &= \frac{1}{2} \rho V_A^2 S \bar{c} C_{l_{CG}}, \\
M &= \frac{1}{2} \rho V_A^2 S \bar{c} C_{m_{CG}}, \\
N &= \frac{1}{2} \rho V_A^2 S \bar{c} C_{n_{CG}}.
\end{align*}
\] (3.3)

In the HIRM problem [2] and McRuer-Ashkenas-Graham [40], the mean aerodynamic chord \( \bar{c} \) is replaced by the wing span \( \delta \) in the formulas for \( L \) and \( N \). In [63] the mean aerodynamic chord \( \bar{c} \) in (3.3) is replaced by a reference length \( \ell \), which is not specified. These modifications of constants in (3.3) are not important for the mathematical structure of the model.

The density of air \( \rho \) is a function of the height \( h \). Formulas for \( \rho \) can be found in the HIRM problem [2] and Barman-Erzberger [10]. When the height of the aircraft does not vary much, \( \rho \) is often taken constant (assumption 11), see e.g. the RCAM problem [1].
Assumption 10 of the modelling assumptions concerns the formulas for the aerodynamic coefficients $C_X$, $C_Y$, $C_Z$, $C_{l_{OG}}$, $C_{m_{OG}}$, and $C_{n_{OG}}$ in (3.3). One can find various modelling equations for these coefficients in the literature. Examples of formulas for the aerodynamic coefficients can be found in Wise-Godhwani-Gu-Dierks-Kerkemeyer-Tang [63], the RCAM problem [1], the HIRM problem [2], and Huang-Tylock-Martorella-Knowles [25]. The aerodynamic coefficients are functions of the variables $\alpha$, $\beta$, Mach, $p$, $q$, $r$, and control surface deflections. The aerodynamic coefficients often depend linearly or quadratically on some of these variables. The aerodynamic coefficients in Mulgund-Stengel [42] include a stochastic process, which represents modelling errors. In many papers the formulas for the aerodynamic coefficients are not complete.

3.3.3 Actuator equations

Models for the control surface actuators are often not specified in the literature. In some papers simple models for the control surface deflections are given. In Kim-Kim [30] one may find linear models for the control surface deflections. In Sobel-Shapiro [56] and the RCAM problem [1] the control surface deflections and their rates are to be within specified bounds. In the HIRM problem [2] both low-order linear models and bounds for the control surface deflections are given.

Models for the engine thrust are often not specified in the literature either. It is often specified only that the thrust may vary from 0 to some upper bound $T_{\text{max}}$. A complex thrust model is given in Barman-Erzerberger [10]. A second-order linear model is used in Biezzad-Azzano [12]. In the RCAM problem [1] the thrust model is a first-order linear model for the throttle setting with bounds on the throttle setting and its rate. The thrust model in the HIRM problem [2] is second-order linear with similar constraints as in the RCAM problem [1].

3.3.4 Wind equations

In Ngo-Ly [43] a sharp-edged gust model of the atmospheric turbulence is described. Also the von-Karman gust model for atmospheric turbulence is presented. Comparing von-Karman with Dryden gust models, it is stated that the von-Karman spectrum provides a better representation of the spectrum obtained from records of true atmospheric turbulence. In the RCAM problem [1] and the HIRM problem [2] the Dryden spectra are used as velocity spectra for turbulence modelling. In Mulgund-Stengel [42] a stochastic model of wind is used to model wind shear.
3.3.5 Sensor equations

The sensors are often assumed to be ideal, i.e., the measurement errors are zero. Exceptions are the HRIM problem [2] in which the sensors are modelled by low-order transfer functions and Mulgund-Stengel [42] in which measurement noise is modelled. In many papers the sensors provide measurements of the state in the general form of the model.

3.4 Mathematical structure of the model

In this section the general forms of the models of aircraft which appear in the literature, are described. Each input, state, and output space is a (subset of a) finite-dimensional Euclidean vector space over $\mathcal{R}$. The models contain two types of inputs:

- control inputs, denoted by $u$, and
- exogeneous inputs, denoted by $w$.

The control inputs are the variables that are available for control by the controller. The exogeneous inputs represent the impact of the environment on the state of the aircraft. The models contain two types of outputs as well:

- measurement outputs, denoted by $y$, and
- regulated outputs, denoted by $z$.

The measurement outputs are the outputs of the sensors, which are available for use by the controller. The regulated outputs are the variables that have to be regulated by the controller. Besides the inputs and outputs the models contain state variables, denoted by $x$, which include the state of the aircraft. The general forms of the models are described in the following four paragraphs. The last paragraph of the section concerns the general form of the constraints on the variables in the model.

Nonlinear deterministic system

A general nonlinear deterministic system is of the form:

$$
\begin{align*}
\dot{x} &= f(x, u, w), \\
y &= g(x, u), \\
z &= h(x), \\
x(t_0) &= x_0,
\end{align*}
$$

(3.4)

with $t \geq t_0$, where $t_0$ is the starting time. No special properties of the functions $f$ and $h$ are mentioned in the literature. The measurement output $y$ is equal to the state $x$ in the literature which has been studied.
Affine nonlinear deterministic system

In many papers the input appears affine (sometimes called linearly in the control) in the state equation of (3.4). Thus the model is an affine nonlinear deterministic system

\[
\begin{align*}
\dot{x} &= a(x) + b(x)u + e(x)w, \\
y &= c(x) + d(x)u, \\
z &= h(x), \\
x(t_0) &= x_0,
\end{align*}
\] (3.5)

with \( t \geq t_0 \). The measurement output \( y \) is equal to the state \( x \) in the literature that has been studied. In addition, in the paper Singh-Steinberg-DiGirolamo [55] the system (3.5) is square and analytic, has finite relative degrees, and has the property that the strong input-output decoupling problem by regular static state feedback is solvable. To ensure the latter property the state space is a subset \( M \) of \( \mathbb{R}^n \). The complement of \( M \) has measure zero.

Linear deterministic system

The linear systems which appear in the literature are of the form

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ew, \\
y &= Cx + Du, \\
z &= Hx, \\
x(0) &= x_0,
\end{align*}
\] (3.6)

with \( t \geq t_0 \). No special properties are mentioned in the literature. In many papers the system matrix \( A \) and the input matrix \( B \) have special forms

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix},
\]

since the system (3.6) often appears as a series connection of linear systems.

Nonlinear stochastic system

The nonlinear stochastic systems that have been found in the literature consist of two equations: a state equation and a measurement equation. The state equation can be either in discrete-time or in continuous-time. In discrete time the state equation is of the form

\[
x_{k+1} = f(x_k, u_k) + e(x_k, u_k)w_k, \quad k = 0, 1, \ldots, \] (3.7)

where \( w_k \) is Gaussian with mean zero and \( E[w_kw_k^T] = \delta_{j,k}Q_k \) for all \( j, k \). The state equation in continuous-time is of the form

\[
\dot{x}(t) = f(x(t), u(t)) + e(x(t), u(t))w(t), \quad t \geq t_0, \] (3.8)
where $w(t)$ is Gaussian with mean zero and covariance function $E[w(t)w(s)^T] = Q(t) \delta(t-s)$. Here $\delta$ is the Dirac delta function (or, in more rigorous mathematical terminology, the Dirac delta distribution, see Schwartz [53]). For the precise mathematical meaning of (3.8) we refer to Chapter 7 of Bagchi [9], where (3.8) is explained in terms of an Itô-integral equation when the state equation is linear and without control input. Following Bagchi [9] it is preferred to write (3.8) as

$$dx_t = f(x_t, u_t)dt + e(x_t, u_t)d\bar{w}_t, \ t \geq t_0,$$

(3.9)

where $\{\bar{w}_t, t \geq 0\}$ is a Brownian motion process with intensity $\Sigma$.

The measurement equation can also be either in discrete or in continuous-time. The measurement equation in discrete-time is given by

$$y(t_k) = c(x(t_k), u(t_k)) + v_k, \ k = 0, 1, \ldots,$$

(3.10)

where $v_k$ is a Gaussian random variable with mean zero and $E[v_kv_j^T] = \delta_{j,k}R_k$ for all $j, k$. For a continuous-time version of (3.10) we refer to Chapter 9 of Bagchi [9].

For the combination of (3.7) and (3.10) the additional assumption $E[w_kv_j^T] = 0$ for all $j, k$ can be found in the literature. For the combination of (3.8) and (3.10) the additional assumption $E[w(t)v_k^T] = 0$ for all $t$ and $k$, can be found.

Constraints

Constraints are of the form

$$k_{\text{min}} \leq k(x, u, \dot{u}) \leq k_{\text{max}},$$

(3.11)

where $k_{\text{min}}$, $k_{\text{max}}$, and $k(x, u, \dot{u})$ are vectors and the inequality signs in (3.11) should be understood componentwise. Constraints on the state appear in Lu-Pierson [35]. Constraints on $u$ and $\dot{u}$ appear often, e.g., in Sobel-Shapiro [56], in the RCAM problem [1], and in the HRM problem [2].
4 Results on optimization problems in flight control design

4.1 Optimization problems

Three classes of optimal flight control problems for dynamical systems appear in the literature:

- the class of optimal tracking problems,
- the class of trajectory optimization problems, and
- the class of optimal state estimation problems.

In the following paragraphs representative problems from the above classes will be stated.

The data for an optimal tracking problem are a deterministic system and a reference trajectory for the regulated output. The optimal tracking problem is to find controllers (or a controller) such that the regulated output of the closed loop system follows the reference trajectory as well as possible. A representative of this class, which covers the optimization problems in several papers, is the optimal output tracking problem by measurement feedback. The data for the problem are a nonlinear deterministic system and a reference trajectory in the space of regulated outputs. The optimal output tracking problem by measurement feedback is to find a measurement feedback control law \( u = k(y) \) of the measurement output such that the regulated output of the closed loop system follows the reference trajectory \( r \) as well as possible.

The following versions of the optimal output tracking problem by measurement feedback appear in the literature.

- The reference trajectory is generated by a system. In this case the tracking problem is called a model-following problem.
- The (full) state is assumed to be measurable. In this case the tracking problem is called the optimal output tracking problem by state feedback.
- The (full) state is assumed to be measurable and to be regulated. In this case the tracking problem is called the optimal state tracking problem by state feedback.

Besides the above versions of the optimal output tracking problem by measurement feedback one may also distinguish between problems for linear and problems for nonlinear systems. In the studied literature the (full) state is assumed to be measurable. Furthermore the problem appears for both linear and nonlinear systems.

The data for a typical trajectory optimization problem are a nonlinear deterministic system with initial conditions on the state and final conditions on the regulated output at some
unspecified final time $t_f$. The problem is to find a final time $t_f$ and a (open loop) control $u$ such that the regulated output at time $t_f$ satisfies the final condition and some optimality is attained. The optimality condition may concern minimum flight time, minimum fuel consumption, or minimum loss of height. A combination of minimum flight time and minimum fuel consumption also appears in the literature. Minimum loss of height is used as an optimality condition in the problem of recovery from wind shear. The trajectory optimization problem has several versions, which are obtained by making the following changes to the previous statement of the problem:

- the final time tag $t_f$ is specified,
- there are no final conditions on the regulated output,
- the regulated output is the state.

The data for the optimal state estimation problem are a stochastic dynamical system, an initial estimate of the state, and at each time tag $t$ the past measurement outputs and the control history. The optimal state estimation problem is to estimate the state based on the past measurement outputs and the control history as well as possible. The optimal state estimation problem has several versions which depend on the state and the measurement equation:

- optimal state estimation problem with continuous-time states and discrete-time measurement outputs,
- optimal state estimation problem with discrete-time states and continuous-time measurement outputs,
- optimal state estimation problem with continuous-time states and continuous-time measurement outputs,
- optimal state estimation with discrete-time states and discrete-time measurement outputs.

Besides the above classification into continuous-time and discrete-time states and measurement outputs, one may also distinguish between optimal state estimation problems with linear and nonlinear dynamical systems.

4.2 Criterion functions and constraints

The optimality conditions for the optimization problems in the previous section are defined in the literature in terms of criterion functions to be minimized or maximized and in terms of constraints to be satisfied. Before presenting the various criterion functions and constraints which appear in the literature, we shall first discuss the role that the
optimality conditions play in solving the optimization problems. This discussion will lead to a reduction of the number of optimality conditions to be considered in this report.

The above stated optimization problems can be solved in two ways:

- to find solutions of the optimization problem and to evaluate afterwards the optimality conditions in order to see whether or not the solutions satisfy the optimality conditions,

- to find solutions of the problem that satisfy the optimality conditions on beforehand.

In this report only the second way of using optimality conditions is considered. In the paragraphs that follow, the corresponding optimality conditions are described as they are found in literature. From the larger class of optimality conditions that are used for evaluation of solutions only, one may find extensive examples in the RCAM and HIRM design challenge reports [1] and [2], respectively.

The following types of criterion functions for optimal tracking problems appear in the literature: the criterion function is

- an integral in which the integrand is a quadratic function of input and regulated output:

\[ J = \int_{t_0}^{\infty} \left( z(t)^T Q(t) z(t) + u(t)^T R(t) u(t) \right) dt, \]

for tracking of \( \tau = 0, \)

- a sum of a quadratic function and an integral in which the integrand is a quadratic function,

- a quadratic function.

The unconstrained optimization problem associated with the first criterion function is to find a state feedback control for a linear system of the form (3.6) such that \( J \) is minimal. This optimization problem is called the LQR (Linear Quadratic Regulator) Problem. It appears in Biezad-Azzano [12], Vincent-Emami Naeini-Khraishi [60], and Wise-Godhwani-Gu-Dierks-Kerkemeyer-Tang [63]. Notice that minimisation of \( J \) leads to both small regulated outputs and small control inputs in the sense of weighted \( L_2 \)-norms. Associated with a criterion function of the second type, both an unconstrained and a constrained optimization problem are solved in Kim-Kim [30]. The unconstrained problem is to find a feedforward gain matrix which minimizes the criterion function. The constrained problem is to find a feedback gain matrix and a feedforward gain matrix such that the closed loop matrix associated with the problem is asymptotically stable and the criterion function is minimized. The third type of criterion functions appears in Singh-Steinberg-DiGirolamo
[35], Khan-Lu [28], Lu-Pierresn [35]. The formulas for the quadratic criterion function in these papers are strongly associated with the predictive control method, which is used to introduce the criterion function. Both constrained and unconstrained versions of the optimization problem appear. The unconstrained optimization problem is to find at each time tag t a control vector u which minimizes the quadratic criterion function. The constrained optimization problem is the unconstrained problems with the constraint that the control vector u(t) is between some specified control bounds.

The criterion functions and constraints for the trajectory optimization problems in the literature that has been studied, are as follows. Given is a nonlinear system with initial value of the form

\[ \dot{x} = f(x, u, t), \quad t \geq t_0, \quad x(t_0) = x_0. \quad (4.1) \]

The criterion function is of the form

\[ J(u) = S(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt, \quad (4.2) \]

for sufficiently smooth functions S and L. In the literature that has been studied (i.e., Barman-Erzerberger [10], Rao-Mees [49], Erberger-Lee [17], Zhao [65]), the criterion function models the minimum flight time only or a weighted average of minimum flight time and fuel consumption. Because of the minimum flight time objective the final time t_f is free. The final state is fixed or constraints on the final state of the form \( \psi(x(t_f), t_f) = 0 \) appear. The optimization problem is to find a final time t_f and a control u on the time interval \([t_0, t_f]\) for which

\[ \min_{u \in \mathcal{U}, t_f \geq t_0} J \]

is attained. The set \( \mathcal{U} \) of admissible controls is not described in the literature. In Zhao [65] an overview of trajectory optimization methods is given. According to Zhao [65] the general form of the criterion function J is given by (4.2). It is not clear whether or not the final time may be fixed or free. In addition to the constraints mentioned above, Zhao [65] also allows path constraints of the form \( \phi = 0 \) or \( \phi \leq 0 \). The path constraints are not specified further.

In Wang [61] the setting of the standard \( H_\infty \)-optimization problem has been used for trajectory optimization. For a linear system \( \Sigma \) of the form

\[ \Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + E_1w(t), \\ y(t) &= Cx(t) + D_1u(t) + E_2w(t), \quad t \geq 0, \\ z(t) &= Hz(t) + D_2u(t) + E_3w(t), \end{cases} \]

the standard \( H_\infty \) optimization problem is to find a linear dynamic feedback compensator \( \Gamma \)

\[ \Gamma : \begin{cases} \dot{x}_c &= A_c x_c + B_c y, \\ u &= C_c x_c + D_c y, \quad t \geq 0, \end{cases} \]
such that the closed loop system is asymptotically stable and the $H_\infty$-norm of the closed loop transfer function $T_{zw}$ from $w$ to $z$ is minimized. Here the $H_\infty$-norm $\|T_{zw}\|_\infty$ of $T_{zw}$ is given by

$$\|T_{zw}\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(T_{zw}(j\omega)),$$

where $\sigma_{\max}(T_{zw}(j\omega))$ is the largest singular value of the matrix $T_{zw}(j\omega)$. The form of the system $\Sigma$ is a little bit more general than the form in Wang [61]. Figure 4.2 shows the set-up for the $H_\infty$-optimization problem.

![Figure 4.1 Set-up for the $H_\infty$ optimization problem](image)

For optimal state estimation problems a criterion function for discrete-time states and discrete-time measurement outputs (see Section 3.3) can be introduced as follows. Following Chapter 3 of Bagchi [9], assume that random vectors $y_1, \ldots, y_k$ are given. These random vectors may be interpreted as measurements of random (state) vectors $x_1, \ldots, x_k$.

The criterion function $J$ is given by

$$J(z) = E[(x_k - z)(x_k - z)^T],$$

where $z$ is any random variable which depends on $y_1, \ldots, y_k$ only and satisfies some additional mathematical properties, which are not important here. The criterion function $J(z)$ has to be minimized in the usual ordering of symmetric matrices, i.e., $J(z) \leq J(z')$ if and only if $J(z') - J(z)$ is positive semi-definite. An estimator $\hat{x}_k$ of $x_k$ is called optimal or a minimum variance estimator of $x_k$ if $J(z)$ is minimal for $z = \hat{x}_k$. A similar definition can be found in Chapter 9 of Bagchi [9] for continuous-time states with continuous-time measurement outputs.
5 Results on optimization methods and techniques in flight control design

5.1 Introduction

To solve the optimization problems that have been stated in the Section 4.2 in terms of criterion functions and constraints, various optimization methods and techniques are applied in the literature. In addition, the literature reports on the software tools that were used. Optimization methods and techniques are presented in Section 5.2; software used for optimization in Section 5.3.

5.2 Methods and techniques

The methods and techniques involve optimization problems for which the criterion function is

- a sum of a (possibly zero) quadratic function and an integral in which the integrand is a quadratic function,
- a quadratic function of a vector,
- a quadratic function of a random variable,
- a nonlinear function of the final state and an integral for which the integrand is nonlinear. More precisely, the criterion function \( J \) is of the form (4.2).

When the criterion function is a sum of a (possibly) zero quadratic function and an integral in which the integrand is a quadratic function, then an explicit solution may be found. The explicit solution may involve solutions of Riccati or Lyapunov equations. Solutions of Riccati equations appear for instance in the solution of the LQR problem and the \( H\infty \)-optimization problem.

For optimization problems with a quadratic criterion function of a vector often an explicit solution is given, when there are no constraints. In this case, the explicit solution is a formula, which may be implemented directly.

When the optimization problem with a quadratic criterion function of a vector is constrained by control bounds on the control \( u(t) \) at each time tag \( t \), i.e.,

\[
L \leq u(t) \leq U, \quad t \geq t_0,
\]

where the inequalities are entrywise, then two approaches are found in the literature:

- in Khan-Lu [28] (see Chapter 8) the unconstrained optimization problem is solved with optimal solution \( u^*(t) \in \mathbb{R}^m \) and the control \( u(t) \) is taken to be \( a(u^*(t)) \), where
\[ \sigma : \mathcal{R}^m \to \mathcal{R}^m \] is the saturation function given by

\[ \sigma(u) = \begin{cases} 
U & \text{if } u \geq U, \\
u & \text{if } L \leq u \leq U, \\
L & \text{if } u \leq L,
\end{cases} \]

- In Lu-Piersson [35], the unconstrained optimization problem is reduced to solving for each time tag \( t \) a nonlinear equation in \( u = u(t) \) of the form \( u = \rho(u) \). For the constrained optimization problem the equation \( u = \sigma(\rho(u)) \) is solved. Referring to Lu [34], it is stated that the latter equation has a unique solution \( u^* \). The solution \( u^* \) is approximated by the iterative scheme \( u_k = \sigma(\rho(u_{k-1})) \), \( k = 1, 2, \ldots \). In this scheme \( u_k \) converges to \( u^* \) whenever the initial value \( u_0 \) is sufficiently close to \( u^* \). It is stated in Lu-Piersson [35] that the iterative scheme "is particularly well suited for digital computers" and that the scheme "converges usually in just a few cycles". It is not explained how the initial value \( u_0 \) should be chosen.

The minimum-variance estimator in optimal state estimation problems is unique (see, e.g., Bagchi [9]). For the case of discrete-time states and discrete-time measurement outputs the minimum-variance estimator \( \hat{x}_k \) is given by the conditional expectation: \( \hat{x}_k = E[x_k|y_1, \ldots, y_k] \). For continuous-time states and continuous-time measurement outputs a similar conditional expectation for the optimal estimator can be found in Bagchi [9]. So the solution of the optimization is reduced to the computation of a conditional expectation. For a linear dynamical system (with a linear measurement equation), the Kalman filter algorithm carries out the computation. The Kalman filter algorithm is attractive for implementation in software, since it is a recursive algorithm. For the nonlinear case, algorithms appear in the literature which look like the Kalman filter algorithm, are recursive, but do not compute the conditional expectation. Nevertheless, such an algorithm may have a good performance. An overview of such algorithms may be found in several books, for instance Gelb [20]. An example of its application in flight control design can be found in Chapter 7.

In Zhao [65] an overview is given of numerical methods and techniques that are used to solve optimization problems associated with the criterion function \( J \) of the form (4.2) and the constraints described in the discussion of trajectory optimization problems in Chapter 4. The following three approaches to optimize \( J \) are mentioned:

- the functional optimization approach,
- the approach by solution of algebraic equations,
- the mathematical programming approach.
Besides the three approaches parameter optimization schemes are mentioned. The associated numerical methods and techniques which are presented in Zhao [65], are:

**Parametric optimization schemes**

- steepest descent
- Newton's method
- conjugate gradient
- variable-metric
- penalty methods
- linear programming
- sequential linear programming
- feasible directions
- reduced gradient method
- sequential quadratic programming

**Functional optimization methods**

- steepest descent,
- conjugate gradient,
- variable-metric method without exact line searches,
- impulse response method
- feasible direction algorithm
- second-order gradient algorithms: like transition-matrix algorithm and sweep method,
- Min-H algorithms,
- differential dynamic programming method,
- quasi-linearisation approach,
- generalised Newton-Raphson Operator method,
- sequential gradient restoration algorithm.
Equation-solving methods

- extremal field method,
- dynamic programming method,
- neighboring extremal algorithm (direct shooting),
- multiple shooting.

Methods through the mathematical programming approach

- Ritz method
- methods on control parametrization,
- methods on state parametrization,
- methods on both state and control parametrization,
- sequential quadratic programming
- accelerated gradient projection technique of parameter optimization

Several methods do not optimize the criterion function $J$ but they solve an associated Hamiltonian two-point boundary value problem. The Hamiltonian two-point boundary value problem is derived from the optimization problem through the Pontryagin minimum principle (also called the Pontryagin maximum principle).

The Pontryagin minimum principle is a general mathematical tool which applies to optimization of the criterion function $J$ in (4.2) with various types of constraints. The various theorems that follow from the principle have the same structure. To see the structure of such theorems, consider the optimization problem associated with the nonlinear system

$$
\dot{x} = f(x, u), \quad t \geq t_0, \quad x(t_0) = x_0,
$$

with criterion function

$$
J(u) = S(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t)) dt,
$$

with fixed final time $t_f$, and without further constraints. Let the Hamiltonian $H$ be given by $H(x, \xi, u) = \xi^T f(x, u) + L(x, u)$. Suppose that $u^*$ is the optimal solution of
the optimization problem. According to Theorem 6 in Nijmeijer [44] there exist vector functions \( x^* \) and \( \xi^* \) on \([t_0, t_f]\) such that

\[
\begin{align*}
\dot{x}^* &= \frac{\partial H}{\partial x}(x^*, \xi^*, u^*)^T, \\
\dot{\xi}^* &= -\frac{\partial H}{\partial x}(x^*, \xi^*, u^*)^T \\
x^*(t_0) &= x_0, \\
\xi^*(t_f) &= -\frac{\partial S}{\partial x}(x^*(t_f))^T,
\end{align*}
\]

(5.1)

where

\[
 u^*(t) = \arg \min_{v \in U} H(x^*(t), \xi^*(t), v), \text{ for all } t \in [t_0, t_f],
\]

(5.2)

with \( U \) the set of admissible controls. The problem to find vector functions \( x^* \) and \( \xi^* \) on \([t_0, t_f]\) that satisfy (5.1), where \( u^* \) is given by (5.2), is called the Hamiltonian two point boundary value problem. The first equation in (5.1) says that \((x^*, u^*)\) is a solution of the system (4.1). The variable \( \xi^* \) is called the costate variable or the adjoint variable. It should be noticed that Theorem 6 in Nijmeijer [44] gives only necessary conditions. In Zhao [65] a more general result than Theorem 6 in Nijmeijer [44] is described. The result in Zhao [65] allows for terminal constraints of the form \( \psi(x(t_f), t_f) = 0 \) and for path constraints of the form \( \phi = 0 \) or \( \phi \leq 0 \). However, a (reference to a) proof of the result is not supplied.

In Rao-Mease [49] the Pontryagin minimum principle has been used for optimization problems with variable final time \( t_f \).

### 5.3 Software

In the literature, several references to software are made, which is used to solve the optimization problems. In this section a short overview of this software is given.

PROTO/OPT is a computer-aided control system design tool, which has been used in Reilly-Eadan-Huang-Levine [50]. Built in is the optimization programme CONSOLE (see Fan-Wang-Koninckx-Tits [18]). CONSOLE can be used for fine-tuning the control law. The optimization programme includes a modified feasible-direction algorithm based on sequential quadratic programming. By the modification, feasible iterates are generated (see Panier-Tits [46]).

In Ly [36] the tool SANDY has been used. It is described as "numerical software" and, "numerical design tool". SANDY allows for "parameter optimization".

In Kumar-Seywald [33] the software

- POST (Program to Optimize Simulated Trajectories, see Brauer-Cornick-Stevenson [14]),
- OTIS (Optimal Trajectories by Implicit Simulation, without reference),
- TODI (Trajectory Optimization via Differential Inclusion, under development by Kumar and Seywald),
- NPSOL, to solve nonlinear programming problems, see Gill-Murray-Saunders-Wright [21]

is mentioned.

According to Wei [62], ADS (see VanderPlaats [59]), IMSL (see [7]), and MINPACK (see [41]) is good constrained optimization software.

Recently developed optimization software includes ANDECS-MOPS (Multi-Objective Parameter Synthesis, see Grübel-Joos [23]) for feedback-control parameter optimization and GESOP (Graphical Environment for Solving Optimization Problems, see Jänsch-Schnepper-Well [26]) for trajectory optimization.
Part II

Description

of

literature
6 Overview of Part II

The present part of the report contains reviews of some papers from the literature on flight control design. The review structure has been developed in detail in Chapter 2. A review of a paper is typically given in the form described in the table below. Conclusions from the reviews are drawn in Chapter 3 on models, in Chapter 4 on optimization problems, and in Chapter 5 on optimization methods and techniques.

Chapter x  <Optimization by optimization method/technique>

Main reference: <main reference>
  x.1  Model of the aircraft
  x.1.1 Modelling assumptions
  x.1.2 Modelling equations
  x.1.2.1 Equations of motion
  x.1.2.2 Aerodynamic equations
  x.1.2.3 Actuator equations
  x.1.2.4 Wind equations
  x.1.2.5 Sensor equations

  x.2  Mathematical structure of the model
  x.2.1 Definitions
  x.2.1.1 State vector for the model of the aircraft
  x.2.1.2 Control inputs
  x.2.1.3 Exogeneous inputs
  x.2.1.4 Measurement outputs
  x.2.1.5 Regulated outputs
  x.2.2 General form of the model

  x.3  Optimization problem
  x.3.1 Data
  x.3.2 Assumptions
  x.3.3 Statement of the optimization problem in words

  x.4  Mathematical formulation of the optimization problem
  x.4.1 Criterion
  x.4.2 Constraints
  x.4.3 The approximate optimization problem: mathematical formulation
  x.4.4 The approximate optimization problem: relation with the original optimization problem
  x.4.5 Design parameters in the optimization problem
x.5 Optimization method or technique
  x.5.1 Further assumptions
  x.5.2 Description of the method or technique
  x.5.3 Further design parameters
  x.5.4 Choice of the design parameters

x.6 Remarks

x.7 Further references

x.8 Comments
7 Optimal state estimation problem by continuous-discrete extended Kalman filter

Main reference: Mulgund-Stengel [42]

7.1 Model of the aircraft

7.1.1 Modelling assumptions

The aircraft is a twin-jet transport aircraft with a gross weight of 38,500 kg and a maximum take-off thrust of 107,000 N. The modelling assumptions that are mentioned in the paper are:

- the flight of the aircraft is in a vertical plane over a flat earth,
- the aircraft is taking off,
- wind shear is present,

7.1.2 Modelling equations

7.1.2.1 Equations of motion

The equations of motion are given by the following nonlinear system:

\[
\begin{align*}
\dot{x} &= V_A \cos \gamma + W_{EB}, \\
\dot{h} &= V_A \sin \gamma - W_{EB}, \\
\dot{V}_A &= \frac{T}{m} \cos \alpha - \frac{D}{m} - g \sin \gamma - \dot{W}_{EB} \cos \gamma + \dot{W}_{EB} \sin \gamma, \\
\dot{\gamma} &= \frac{1}{V_A} \left( \frac{T}{m} \sin \alpha + \frac{L}{m} - g \cos \gamma + \dot{W}_{EB} \cos \gamma + \dot{W}_{EB} \sin \gamma \right), \\
\dot{\alpha} &= q - \dot{\gamma}, \\
I_{\phi,\dot{q}} &= M.
\end{align*}
\]

(7.1)

7.1.2.2 Aerodynamic equations

The lift force \( L \), the drag force \( D \), and the pitching moment \( M \) are modelled by the equations

\[
\begin{align*}
D &= \frac{1}{2} \rho S C_D + m \, w_{f1}, \\
L &= \frac{1}{2} \rho S C_L + mV_A \, w_{f2}, \\
M &= \rho S \bar{C}_M.
\end{align*}
\]

(7.2)

The symbols \( \rho, S, C_D, C_L, C_M \) and \( \bar{C} \) are not specified. The variables \( w_{f1} \) and \( w_{f2} \) represent uncertainty in the aerodynamic model. The vector \( w_f = \left( w_{f1} \ w_{f2} \right) \) is modelled as Gaussian white noise with mean zero and covariance function \( E[w_f(t)w_f(\tau)^T] = \delta(t - \tau)Q_f(t) \), where \( \delta \) denotes the Dirac delta function (or, in more rigorous mathematical terminology, the Dirac delta distribution, see Schwartz [53]).
7.1.2.3 Actuator equations

The engine thrust force $T$ is modelled by

$$\dot{T} = \frac{1}{2}(T_c - T).$$

(7.3)

7.1.2.4 Wind equations

The impact of wind shear on the motion of the aircraft is modelled using the techniques described in Frost-Bowles [19] and Stengel [57]. The resulting model of wind shear is stochastic. Let $W_E^{(3)}$ be the vector of the third derivatives of $W_{xB}$ and $W_{zB}$ with respect to time $t$. The vector $W_E^{(3)}$ is taken to be zero mean Gaussian white noise with

$$E[W_E^{(3)}(t)W_E^{(3)}(\tau)^T] = Q(t)\delta(t - \tau),$$

(7.4)

where $Q(t)$ is a positive semidefinite diagonal matrix of order 2.

7.1.2.5 Sensor equations

The sensor outputs are $h, V, V_A, \alpha, \theta, q, h, \dot{h}, z, \dot{z}$. The sensors are assumed to give noisy measurement outputs in discrete-time. The frequency of the measurements is not specified. The noises of the different sensors are assumed to be independent of each other. The noise of the $i$-th sensor is modelled as a Gaussian white noise process with zero mean and with variance $\sigma_i$ depending on the sensor. Put

$$y_{\text{true}} = (h, V, V_A, \alpha, \theta, q, \dot{h}, \dot{z}, \ddot{h})^T.$$

(7.5)

So the sensor equations are

$$y(t_k) = y_{\text{true}} + v_k, \quad k = 0, 1, \ldots,$$

(7.6)

where the measurement noise $\{v_k\}$ is zero mean Gaussian white noise that is uncorrelated with the exogeneous input:

$$E[v_kv_k^T] = R_k,$$

$$E[v_kw_j(t)^T] = E[v_kW_E^{(3)}(t)^T] = 0, \quad \text{for all } k, t$$

where $R_k = \text{diag}(\sigma_i)_{i=1}^n$. The function $h$ and the discrete-time tags $t_k$ in (7.6) are not specified.

7.2 Mathematical structure of the model

7.2.1 Definitions

7.2.1.1 State vector for the model of the aircraft

The state vector $x$ is given by

$$x = (x, h, V_A, \gamma, \alpha, q, T, W_{xB}, W_{zB}, \dot{W}_{xB}, \dot{W}_{zB}, \ddot{W}_{xB}, \ddot{W}_{zB})^T.$$

(7.7)
Notice that the notation, which has been taken from p. 40 of the communication handbook of FM(AG08) ([3]) is ambiguous: the first $x$ in (7.7) denotes the state; the second $x$ the $x$-position of the centre of gravity of the aircraft.

7.2.1.2 Control inputs

The control inputs are the commanded thrust $T_c$ and the elevator deflection $\delta_E$. The impact of the commanded thrust on the engine thrust $T$ is modelled by (7.3). The impact of the elevator deflection on the behaviour of the aircraft is not specified. Put $u = (T_c, \delta_E)^T$.

7.2.1.3 Exogeneous inputs

The exogeneous inputs are $W_E^{(3)}$, which models the third derivative of the wind field, $\{v_k\}$, which models the sensor noises, and $w_f$, which models the uncertainty in the model. Thus the exogeneous input $w$ is given by $w = (0, w_f^T, (W_E^{(3)})^T)^T$.

7.2.1.4 Measurement outputs

Measurements $y$ are taken from the vector $y_{\text{true}}$ given by (7.5).

7.2.1.5 Regulated outputs

not applicable

7.2.2 General form of the model

The general form of the model is

$$
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) + w(t), \\
y(t_k) &= c(x(t_k)) + v_k,
\end{align*}
$$

(7.8)

The stochastic process $w$ is called the system noise; the stochastic process $\{v_k\}$ the measurement noise. The stochastic processes $w(t)$ and $\{v_k\}$ are white Gaussian processes with zero mean and

$$
\begin{align*}
E[w(t)w(s)^T] &= \delta(t-s)Q_c(t), \quad t, s \geq 0, \\
E[v_jv_k^T] &= \delta_{j,k}R_k, \quad j, k = 0, 1, \ldots, \\
E[w(t)v_k^T] &= 0, \quad t \geq 0, \quad k = 0, 1, \ldots.
\end{align*}
$$

7.3 Optimization problem

7.3.1 Data

The data for the optimization problem are the equations (7.8), initial data, measurement outputs, and control inputs. More precisely, the data are the equations (7.8),
• the expected values \( \hat{x}(t_0) \) and \( P_0 \) of the initial state and the covariance matrix of the initial state, respectively, i.e.,
\[
\hat{x}(t_0) = E[x(t_0)], \quad P_0 = E[(x(t_0) - \hat{x}(t_0))(x(t_0) - \hat{x}(t_0))^T],
\]
and at each time tag \( t \) with \( t_k \leq t < t_{k+1} \):
• the measurement outputs \( y(t_0), y(t_1), \ldots, y(t_k) \),
• the control history \( u(t'), 0 \leq t' \leq t \).

In the paper the control history is not mentioned explicitly and the initial data \( \hat{x}(t_0) \) and \( P_0 \) are not specified.

7.3.2 Assumptions

It should be assumed that the covariance matrix \( R_k \) is positive definite for any \( k \). This assumption is not stated in the paper.

7.3.3 Statement of the optimization problem in words

The optimization problem is to find at each time tag \( t \) with \( t_k \leq t < t_{k+1} \) an estimator \( \hat{x}(t_k) \) of the state \( x(t_k) \) on the basis of the measurement outputs \( y(t_0), y(t_1), \ldots, y(t_k) \) such that the estimator of the stochastic process is optimal. For a discussion of the meaning of the phrase "optimal" is referred to Section 7.8.

7.4 Mathematical formulation of the optimization problem

7.4.1 Criterion

The paper does not contain a criterion function. For a discussion about a criterion function, see Section 7.8 and Chapter 5.

7.4.2 Constraints

not applicable.

7.4.3 The approximate optimization problem: mathematical formulation

not applicable.

7.4.4 The approximate optimization problem: relation with the original optimization problem

not applicable.
7.4.5 Design parameters in the optimization problem

The design parameters are the spectral density matrices $Q_c(t)$, $t \geq 0$ and the covariance matrices $R_k$, $k = 0, 1, \ldots$. The matrices $Q_c(t)$ should be positive semi-definite for each $t \geq 0$; the matrices $R_k$ positive definite for each $k = 0, 1, \ldots$.

7.5 Optimization method or technique

7.5.1 Further assumptions

Not applicable.

7.5.2 Description of the method or technique

The states are estimated by the (continuous-discrete) extended Kalman filter (EKF). The formulas in the paper only concern the estimators at time tags $t_k$. Integral equations are used in place of a differential equations. The integral formulas in the paper are incomplete. A numerically stable implementation of the formulas is not discussed. More complete formulas for the EKF can be found in a bench of books, e.g., Gelb [20]. Besides estimators of the states, the EKF also provides estimators of the covariance matrices of the estimated states. More precisely, for any time tag $t$ for which no measurement is available (i.e., $t \neq t_k$, $k = 0, 1, \ldots$), the EKF provides a predictor of the state at time tag $t$ and of the corresponding covariance matrix. For any time tag $t_k$, the EKF provides an estimator of the state at time tag $t_k$ and of the corresponding covariance matrix. The predictors and estimators are generated recursively in $k = 0, 1, \ldots$. In the $k$-th recursive step, the data for the predictor are the estimator $\hat{x}(t_{k-1}|t_{k-1})$ of $x(t_{k-1})$, the corresponding covariance matrix $P(t_{k-1}|t_{k-1})$, and the control history $u(t')$, $t_{k-1} \leq t' \leq t_k$. In this step the EKF provides for each $t$ with $t_{k-1} < t \leq t_k$ a predictor $\hat{x}(t|t_{k-1})$ of the state $x(t)$ and a predictor $P(t|t_{k-1})$ of the corresponding covariance matrix. Next the measurement output $y(t_k)$ is used as well to adapt the predictor $\hat{x}(t_k|t_{k-1})$ and $P(t_k|t_{k-1})$ to estimators $\hat{x}(t_k|t_k)$ and $P(t_k|t_k)$ of the state $x(t_k)$ and of the corresponding covariance matrix $P(t_k|t_k)$, respectively. The EKF includes matrix multiplications, inversion of positive definite matrices, and solving initial value ordinary (nonlinear) differential equations.

7.5.3 Further design parameters

No further design parameters.

7.5.4 Choice of the design parameters

In the paper the choice of $R_k$ is based on the state-of-the-art for inertial and air-data systems. The system noise spectral density matrix $Q_c$ is of the form $Q_c = \text{diag}(0, Q_f, Q)$. 
Both $Q_f$ and $Q$ are positive semi-definite diagonal matrices of order 2. To choose the diagonal entries of $Q$, the case when the uncertainty $w_f$ in the aerodynamic equations is identically zero has been considered. The diagonal entries of $Q$ have been chosen by trial and error to find a good trade-off between attenuation of the measurement noise and minimization of the lags in the estimators of the state components. With $Q$ fixed, the diagonal elements of $Q_f$ have been chosen by estimating the effect of an instantaneous 10% increase in lift and drag on $\dot{x}_{ac}$ in a typical steady flight condition. Here $x_{ac} = (x, h, V_A, \gamma, \alpha, q, T)^T$, where the subscript ac stands for aircraft. The diagonal elements of $Q_f$ have first been set equal to the magnitude of the change in $\dot{x}_{ac}$ and then refined to achieve satisfactory performance for the combination of the continuous-discrete Kalman filter and a nonlinear inverse dynamics control law.

7.6 Remarks

In the paper the extended Kalman filter algorithm is evaluated in combination with a nonlinear inverse dynamics control law. More precisely, the performance of a nonlinear inverse dynamics control law based on the exact knowledge of the state is compared with the performance of a nonlinear inverse dynamics control law based on the estimated state, where the estimated state is obtained from the extended Kalman filter algorithm.

7.7 Further references

There exist various methods for various optimal estimation problems. Examples of the rich literature on optimal estimation are the books Gelb [20], Brown-Hwang [15], and Saridis [52]. Recent papers include the MUC filter in Mason-Mook [39], a process noise covariance estimator in Mason-Mook [38], and a geometric nonlinear filter in Bishop-Antoula [13]. Main ideas for a numerically stable implementation of the EKF can be found in Bierman [11] and Grewal-Andrews [22].

7.8 Comments

In the paper the engine thrust $T$ is also modelled by

$$T = T_{max}(V_A)\delta_{TH}, \quad 0 \leq \delta_{TH} \leq 1.$$  \hspace{1cm} (7.9)

Equation (7.9) seems to be superfluous in view of equation (7.3). Furthermore it seems ambiguous that in the paper $Q_c$ is used both as a design parameter to tune the optimization algorithm and as a parameter which indicates the severeness of the wind shear that is encountered by the aircraft.

In the optimal state estimation problem, the meaning of optimal is not stated. This lack of definition of the phrase "optimal" is quite common in theory and application of the EKF, because for the usual mathematical definitions of optimal the EKF is a non-optimal solution of the corresponding optimization problem. However, the EKF has often shown
good performance in practical problems, for instance in the field of navigation. For further discussion on optimal state estimation, see Chapters 4 and 5.
8 Optimal state tracking by predictive control

Main reference: Khan-Lu [28].

8.1 Model of the aircraft

8.1.1 Modelling assumptions

The aircraft model is based on the model for the 1991-1992 AIAA Control Design Challenge in Burumbaugh [16]. The aircraft has a weight of 45,000 lb and a wing area of 608 ft². The aircraft has two turbofan engines capable of producing 34,000 lb of thrust each. The maximum speed is Mach 2.5. The absolute ceiling of the aircraft is approximately 60,000 ft.

8.1.2 Modelling equations

8.1.2.1 Equations of motion

The equations of motion are

\[
\begin{align*}
\dot{x} &= V_A \cos(\theta - \alpha) + W_{xg}, \\
\dot{h} &= V_A \sin(\theta - \alpha), \\
\dot{V}_A &= \frac{1}{m}(T \cos \alpha - D - mg \sin(\theta - \alpha) - \dot{W}_{xg} \cos(\theta - \alpha)), \\
\dot{\alpha} &= \frac{1}{mV_A} \left( -T \sin \alpha - L + mg \cos(\theta - \alpha) - m\dot{W}_{xg} \sin(\theta - \alpha) \right) + q, \\
\dot{\theta} &= q, \\
I_{y\dot{q}} &= M. 
\end{align*}
\]  

(8.1)

The equations (8.1) are in a one-one relation with the equations (7.1). The corresponding relation is \( \gamma = \theta - \alpha \).

8.1.2.2 Aerodynamic equations

The lift force \( L \), the drag force \( D \), and the pitching moment \( M \) are not specified.

8.1.2.3 Actuator equations

The engine thrust \( T \) is not specified.

8.1.2.4 Wind equations

Only horizontal wind is included in the model. The horizontal wind is considered to be a disturbance.
8.1.2.5 Sensor equations

The sensors measure the pitch angle \( \theta \), the height \( h \), the horizontal position \( z \), the air speed \( V_A \), the angle of attack \( \alpha \), and the pitch rate \( q \). The sensors are assumed to be ideal and to give the measurements continuously.

8.2 Mathematical structure of the model

8.2.1 Definitions

8.2.1.1 State vector for the model of the aircraft

The state vector \( z \) is given by \( z = (x_1^T, x_2^T)^T \) with

\[
\begin{align*}
x_1 &= (\theta, h, x)^T, \\
x_2 &= (V_A, \alpha, q)^T,
\end{align*}
\]  

(8.2)

8.2.1.2 Control inputs

The control inputs are the symmetric stabilator deflection \( \delta_H \) and the throttle setting \( \delta_{TH} \). The impact of these inputs on the aircraft is not specified. The control inputs should be between some control bounds, which are also not specified. Put \( u = (\delta_H, \delta_{TH})^T \).

8.2.1.3 Exogeneous inputs

The exogeneous input \( w \) is given by \( w = W_{xe} \).

8.2.1.4 Measurement outputs

The measurement output \( y \) is given by \( y = x \).

8.2.1.5 Regulated outputs

The regulated output \( z \) is given by \( z = x \).

8.2.2 General form of the model

The aircraft model is of the form

\[
\begin{align*}
\dot{x}_1 &= f_1(x) + g_1(x)w(x_1), \\
\dot{x}_2 &= f_2(x) + B(x)u + g_2(x)w(x_1), \\
y &= x, \\
z &= x.
\end{align*}
\]  

(8.3)
8.3 Optimization problem

8.3.1 Data

At each time tag $t$ the data are:

- a reference state trajectory $r(t)$, $t_0 \leq t \leq t_f$,
- the state history $x(t')$, $t_0 \leq t' \leq t$.

In Khan-Lu [28] the reference state trajectory $r$ is a minimum time to climb trajectory. A detailed description of the method used to obtain the reference trajectory is given in Khan-Lu [29].

8.3.2 Assumptions

no assumptions.

8.3.3 Statement of the optimization problem in words

In a general form the state reference tracking optimization problem is to find a control law which is based on the (measured) past states and (measured) actual state, which ensures that the state of the closed loop system follows the reference state trajectory as well as possible.

8.4 Mathematical formulation of optimization problem

8.4.1 Criterion

In the paper two criterion functions are considered. The criterion functions are defined at each time tag $t$ with $t_0 \leq t \leq t_f$. Let $e$ be the state error function, i.e.,

$$
e(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} = x(t) - r(t) = \begin{pmatrix} x_1(t) - r_1(t) \\ x_2(t) - r_2(t) \end{pmatrix}.
$$

(8.4)

Put $t_k = t$ and $e_k = e(t_k)$.

To define the first criterion function at time tag $t = t_k$ we introduce the function

$$
\tilde{J}_{1,k} = \tilde{e}_{k+1,1}^T Q_1 \tilde{e}_{k+1,1} + \Delta^2 \tilde{e}_{k+1,2}^T Q_2 \tilde{e}_{k+1,2},
$$

(8.5)

where $\tilde{e}_{k+1,1}$ and $\tilde{e}_{k+1,2}$ denote the predictions of the errors $e_1(t)$ and $e_2(t)$, respectively, at time $t_{k+1} > t_k$ and $\Delta = t_{k+1} - t_k$. The vectors $\tilde{e}_{k+1,1}$ and $\tilde{e}_{k+1,2}$ in (8.5), which are not known at time $t_k$, are estimated by a second- and first-order Taylor polynomial, respectively, for the case when $w = 0$:

$$
\tilde{e}_{k+1,1} = e_{k+1,1} + \Delta [f_1(x_k) - \hat{r}_1(x_k)] + \frac{1}{2} \Delta^2 [f_{11} f_1(x_k) + f_{12} f_2(x_k) + f_{12} B(x_k) u_k - \hat{r}_1(x_k)],
$$

(8.6)

$$
\tilde{e}_{k+1,2} = e_{k+1,2} + \Delta [f_2(x_k) + B(x_k) u_k - \hat{r}_2(x_k)],
$$
where
\[ F_{11} = \frac{\partial f_1}{\partial x_1}(x_k), \quad F_{12} = \frac{\partial f_1}{\partial x_2}(x_k). \]

The first criterion \( J_{1,k} \) at time \( t_k \) is defined to be the right hand side of (8.5) with \( \hat{e}_{k+1,1} \) and \( \hat{e}_{k+1,2} \) replaced by the right hand side of (8.6).

To define the second criterion function we introduce \( \tilde{J}_{2,k} \) given by
\[ \tilde{J}_{2,k} = \hat{e}_{k+1,1}^T Q_1 \hat{e}_{k+1,1} + \hat{e}_{k+1,1}^T Q_2 \hat{e}_{k+1,1} + \hat{e}_{k+1,2}^T Q_3 \hat{e}_{k+1,2}, \quad (8.7) \]

where \( \hat{e}_{k+1,1} \) (resp., \( \hat{e}_{k+1,1} \) and \( \hat{e}_{k+1,2} \)) denotes the prediction of the error \( e_1(t) \) (resp., the derivative \( \hat{e}_1(t) \) and the error \( e_2(t) \) at time \( t_{k+1} \)). The vectors \( \hat{e}_{k+1,1}, \hat{e}_{k+1,1}, \) and \( \hat{e}_{k+1,2} \) are estimated by Taylor polynomials for the case when \( w = 0 \) in a similar way as in the previous paragraph. When these estimates are substituted in the right hand side of (8.7), then the definition of the second criterion function \( J_{2,k} \) is obtained.

8.4.2 Constraints

Constraints are some unspecified bounds on the control input \( u \).

8.4.3 The approximate optimization problem: mathematical formulation

For each time tag \( t = t_k \) the following two approximate optimization problems are solved.

Problem 1 Given the state \( x_k \) at time tag \( t_k \), the approximate optimization problem is to find the input \( u_k \) at time \( t_k \) which minimizes \( J_{1,k} \).

Problem 2 Given the state \( x_k \) at time tag \( t_k \), the approximate optimization problem is to find the input \( u_k \) at time \( t_k \) which minimizes \( J_{2,k} \).

8.4.4 The approximate optimization problem: relation with the original optimization problem

When the time lag \( \Delta \) is sufficiently small, one may expect that the state of the closed loop system with the solution of Problem 1 is near the reference state. This expectation is not proved to be right in the paper. Similar statements hold true for Problem 2.

8.4.5 Design parameters in the optimization problem

At each time tag \( t \) the design parameters are the time lags \( \Delta = \Delta(t) \) and the weighting matrices \( Q_1 = Q_1(t), Q_2 = Q_2(t), \) and \( Q_3 = Q_3(t) \) in (8.5) and (8.6).
8.5 Optimization method or technique

8.5.1 Further assumptions

8.5.2 Description of the method or technique

For both Problem 1 and Problem 2 an explicit solution is given for the case when the control input \( u \) is unrestricted. If any of the control inputs is saturated, it is set to the corresponding boundary value.

8.5.3 Further design parameters

not applicable

8.5.4 Choice of the design parameters

The design parameters \( Q_1, Q_2 \) and \( Q_3 \) are chosen to be diagonal matrices. By taking zeros on the diagonal the number of controlled states can be determined. It appeared that two or three appropriate (see the paper) controlled states were sufficient in Problem 1. Furthermore the design parameter \( \Delta = \Delta(t) \) has been taken varying in \( t \). It has been optimized within a closed interval by Brent’s algorithm (see pages 361-363 in Kahaner-Moler-Nash [27]). In Problem 2 the design parameter \( \Delta \) has been taken constant with \( \Delta = 1.0 \) s.

8.6 Remarks

The parameter \( \Delta \) should not be interpreted as an integration step size but as a design parameter according to the paper.

The quality of the controller is evaluated through the maximum height error function \( E_{\text{max}} \) and the average height error function \( \bar{E} \) given by

\[
E_{\text{max}} = \max_{t_0 \leq t \leq t_f} |h(t) - h_{\text{ref}}(t)|, \quad \bar{E} = \frac{1}{t_f-t_0} \int_{t_0}^{t_f} |h(t) - h_{\text{ref}}(t)| \, dt,
\]

where \( h_{\text{ref}} \) is the second component of the state reference trajectory \( r \).

Robustness with respect to the external disturbance of wind has been evaluated for the solution of Problem 2 with the wind field

\[
W_{z_f}(h) = \begin{cases} 
80 \cos(h\pi/10000) & \text{if } 0 \leq h \leq 25,000 \text{ ft}, \\
0 & \text{otherwise}.
\end{cases}
\]

Further robustness of the solution to Problem 2 has been evaluated with respect to disturbances on the aerodynamic data and with respect to initial errors.
8.7 Further references

no further references.

8.8 Comments

It seems more natural to replace the coefficient $\frac{1}{t_f}$ in the definition of $\bar{E}$ by $\frac{1}{t_f-t_0}$. Furthermore it should be noticed that robustness is not included in the mathematical formulation of the optimization problem. Finally, for a follow-up study of the paper it is recommended to study whether or not controls get saturated and which is the impact of saturated controls on the closed loop performance.
9 Optimal output trajectory tracking by feedforward/feedback control

Main reference: Kim-Kim [30].

9.1 Model of the aircraft

9.1.1 Modelling assumptions

The aircraft is an AFTI/F-16. Its behaviour is assumed to be approximated by linear models for the short period behaviour of the aircraft and for the control actuator dynamics, both with Mach number $M = 0.6$ and altitude $h = 3000$ ft.

9.1.2 Modelling equations

9.1.2.1 Equations of motion

The equations of motion are given by the linear input/state system

\[
\begin{pmatrix}
\dot{\gamma} \\
\dot{q} \\
\dot{\alpha}
\end{pmatrix} = A_1 \begin{pmatrix}
\gamma \\
q \\
\alpha
\end{pmatrix} + B_1 \begin{pmatrix}
\delta_E \\
\delta_F
\end{pmatrix},
\]  

(9.1)

where the first column of the matrix $A_1$ is zero and $B_1$ is a matrix without zero entries. Complete formulas for $A_1$ and $B_1$ may be derived from the paper.

9.1.2.2 Aerodynamic equations

not applicable

9.1.2.3 Actuator equations

The actuators of the elevator and the flaperon are modelled by the linear input/state system

\[
\begin{pmatrix}
\delta_E \\
\delta_F
\end{pmatrix} = A_2 \begin{pmatrix}
\delta_E \\
\delta_F
\end{pmatrix} + B_2 \begin{pmatrix}
\delta_{E,c} \\
\delta_{F,c}
\end{pmatrix},
\]  

(9.2)

with $A_2 = \text{diag}(-20, -20)$ and $B_2 = -A_2$. The paper has a typing error in $A_2$.

9.1.2.4 Wind equations

Wind is not modelled.
9.1.2.5 Sensor equations

The sensors measure the flight path angle, the pitch rate, the elevator deflection, the flaperon deflection, and an unspecified variable \( y_2 \) that depends linearly on the pitch rate, the angle of attack, the elevator deflection, and the flaperon deflection. From the paper Sobel-Shapiro [56], which contains more details about the model, it can be deduced that \( y_2 \) is the normal acceleration \( a_{z,p} \) at the pilot's station.

9.2 Mathematical structure of the model

9.2.1 Definitions

9.2.1.1 State vector for the model of the aircraft

The state vector \( z \) is given by

\[
x = (\gamma, q, \alpha, \delta_E, \delta_F).
\]

9.2.1.2 Control inputs

The control inputs are the elevator deflection command \( \delta_{E,c} \) and the flaperon deflection command \( \delta_{F,c} \). The impact of the control inputs on the elevator deflection \( \delta_E \) and the flaperon deflection \( \delta_F \) is given by (9.2). Put \( u = (\delta_{E,c}, \delta_{F,c}) \).

9.2.1.3 Exogenous inputs

not applicable

9.2.1.4 Measurement outputs

The measurement output \( y \) is given by \( y = (q, a_{z,p}, \alpha, \delta_E, \delta_F) \).

9.2.1.5 Regulated outputs

The regulated output \( z \) is not specified. It can be deduced from Sobel-Shapiro [56] that the components of \( z \) are the pitch angle and the flight path angle.

9.2.2 General form of the model

The open loop system is of the form

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx \\
z &= Hx
\end{align*}
\]

(9.3)
where the matrix $H$ is not specified and the matrices $A$, $B$, and $C$ are given by

$$A = \begin{pmatrix} A_1 & B_1 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with the $*$s denoting nonzero entries, which are specified in the paper.

### 9.3 Optimization problem

#### 9.3.1 Data

Two optimization problems are solved in the paper. They will be called Problem 1 and Problem 2. The data for the optimization problems are as follows.

*Data 1* The data for Problem 1 are a reference signal $r$ and a feedback controller $K$ such that the closed loop system is asymptotically stable, i.e., the matrix $A_K = A + BKC$ has all eigenvalues in the open left half plane.

*Data 2* The data for Problem 2 is a reference signal $r$.

In the paper, the reference signal is a step input that represents the command of the pilot.

#### 9.3.2 Assumptions

no assumptions.

#### 9.3.3 Statement of the optimization problem in words

The set-up for both optimization problems is presented in Figure 9.1, in which $\Sigma$ is the system given by (9.3).

![Diagram](attachment:image.png)

*Fig.9.1 Set-up for the optimization problems*

Optimization problem 1 is to find a feedforward controller $F$ such that the regulated output $z$ of the feedforward/feedback controlled system follows the reference trajectory $r$ as well as possible.
Optimization problem 2 is to find a (asymptotically) stabilizing feedback controller $K$ and a feedforward controller $F$ such that the regulated output of the feedforward/feedback controlled system follows the reference trajectory $r$ as well as possible.

9.4 Mathematical formulation of the optimization problem

9.4.1 Criterion

For the definition of the criteria for Problems 1 and 2, we need to introduce some notation. Let $e$ be the tracking error between the regulated output and the reference trajectory, i.e., $e = z - r$. Let $\tilde{r}$ be a steady state value of a reference trajectory. Let $\tilde{x}$ be the corresponding steady-state value of the closed loop state equation $\dot{x} = A_K x + B F \tilde{r}$, i.e., $\tilde{x} = -A_K^{-1} B F \tilde{r}$. Let $\tilde{u}$ be the corresponding steady state input, i.e., $\tilde{u} = K C \tilde{x} + F \tilde{r}$. Let $\tilde{e}$ be the corresponding value of the tracking error, i.e., $\tilde{e} = H \tilde{x} - \tilde{r}$. Put

$$
\tilde{e} = e - \bar{e}, \quad \tilde{u} = u - \bar{u}.
$$

In words, $\tilde{e}$ and $\tilde{u}$ are the deviations of the tracking error and the input from their steady state values.

**Problem 1** The criterion function for Problem 1 is

$$
J(F, \tilde{r}) = \int_0^\infty (\tilde{e}^T Q \tilde{e} + \tilde{u}^T R \tilde{u}) \, dt + \tilde{e}^T V \tilde{e}.
$$

**Problem 2** The criterion function is

$$
J(K, F, \tilde{r}) = \int_0^\infty (\tilde{e}^T Q \tilde{e} + \tilde{u}^T R \tilde{u}) \, dt + \tilde{e}^T V \tilde{e}. \quad (9.4)
$$

9.4.2 Constraints

**Problem 1** Constraints are the equations (9.3).

**Problem 2** Constraints are the equations (9.3). Furthermore the matrix $A_K = A + B K C$ must have all eigenvalues in the open left half plane.

9.4.3 The approximate optimization problem: mathematical formulation

**Problem 1** Optimization problem 1 is to find the feedforward matrix $F$ that minimises $J(F, \tilde{r})$ for any $\tilde{r}$.

**Problem 2** Optimization problem 2 is to to find a feedback matrix $K$ and a feedforward matrix $F$ that minimise $J(K, F, \tilde{r})$ for any $\tilde{r}$ under the constraint given in Section 9.3.5.

9.4.4 The approximate optimization problem: relation with the original optimization problem

If in any of the two problems the criterion function takes a small value, then both $\tilde{e}$ and $\tilde{e}$ are small. Hence the tracking error $e = \tilde{e} + \bar{e}$ is small. In addition, deviations of $u$ from
the steady state value $\bar{u}$ are small.

9.4.5 Design parameters in the optimization problem

In both optimization problems the weighting matrices $Q$, $R$, and $V$ are positive semi-
definite design parameters with $Q \geq 0$, $R > 0$, and $V \geq 0$.

9.5 Optimization method or technique

9.5.1 Further assumptions

It is assumed that $x(0) = 0$.

9.5.2 Description of the method or technique

Problem 1 The criterion function may be rewritten as

\[
J(F, \bar{r}) = \bar{r}^T \left( F^T B^T A_K^{-1} P_K A_K^{-1} B F + (I + H A_K^{-1} B F)^T V (I + H A_K^{-1} B F) \right) \bar{r},
\]

(9.5)

where $P_K$ is the unique solution of the Lyapunov equation

\[
A_K^T P_K + P_K A_K + H^T Q H + C^T K^T R K C = 0
\]

(9.6)

Thus a feedforward gain $F$ for which $J(F, \bar{r})$ is minimized, is given by

\[
F = - \left( B^T A_K^{-1} (P_K + H^T V H) A_K^{-1} B \right)^{-1} B^T A_K^{-1} H^T V.
\]

(9.7)

Problem 2

Let $f$ be the left hand side of (9.6). The Lagrangian (in the paper called Hamiltonian)
associated with the equations (9.5) and (9.6) is given by

\[
\mathcal{H}(K, F, \bar{r}) = J(K, F, \bar{r}) + \text{trace} (f S),
\]

where $S$ is a symmetric matrix of Lagrange multipliers. To find an optimal feedback gain
matrix $K$ and feedforward gain matrix $F$ the following iterative procedure in 5 steps is
carried out:

1. Choose the initial feedback gain matrix $K_0$ such that the closed loop system matrix
   $A_{K_0}$ is asymptotically stable. Put $K = K_0$.

2. With the given feedback gain matrix $K$ solve the Lyapunov equation (9.6) for $P_K$.

3. With $K$ and $P_K$, define the feedforward gain matrix $F$ by (9.7).

4. With the obtained matrices, compute the gradient $\frac{\partial J}{\partial K}$.

5. If the computed gradient is not sufficiently small, update the feedback gain matrix
   $K$ using the gradient information and go to 2.
It is not specified how the gradient information in step 5 should be used. According to the paper any gradient algorithm can be used for updating the feedback gain matrix $K$. In the study on which the paper reports, the Broydon-Fletcher-Goldfarb-Shanno algorithm (one of the quasi-Newton methods) in the book Press-Flannary-Teukolsky-Vetterling [48] has been used as gradient algorithm. Furthermore it is not specified how the Lagrange multiplier matrix $S$ is updated and how $r$ is handled.

9.5.3 Further design parameters

not applicable.

9.5.4 Choice of the design parameters

The design parameters $Q$, $R$, and $V$ have been found by trial and error. As a first trial the design parameters have been taken scalar multiples of the identity matrix of the appropriate order. Next one of the three design parameters has been fixed and the other scalar multiples have been tuned. Finally, off-diagonal entries in weighting matrices could have been added for further investigation.

9.6 Remarks

The criterion functions in Problems 1 and 2 have the advantage that they make both $\dot{e}$ and $\ddot{e}$ small. Thus both the transient response and the steady-state response are improved. In the paper the reference signal is not fully specified. The flight path angle in the reference signal is constant zero, but the pitch angle is specified by its graph only.

9.7 Further references

In Kim-Kim [31] only Problem 1 is solved. Compared to the treatment of Problem in Kim-Kim [30], the analysis of Problem 1 is extended with an analysis of the limiting cases for the choice of the design parameters. The aircraft model is also used in Sobel-Shapiro [56], in which more detail is provided. Also of interest is the paper Knox-McCarty [32]. In the latter paper the problem to find a stabilising output feedback gain matrix that minimizes a quadratic cost function is discussed. For the solution of this problem two numerical algorithms are given. Furthermore, an algorithm is given for the solution of the problem with additional linear constraints on the entries of the feedback gain matrix.

9.8 Comments

In this section we make five comments about the optimization method used for Problem 2. Firstly, it seems necessary that in Step 5 the updated feedback gain matrix $K$ also has the property that the closed loop state matrix $A_K$ is asymptotically stable. It is not clear
why this property of the updated feedback gain matrix is guaranteed by some gradient algorithm. Notice that when \( A_K \) is not asymptotically stable, then the Lyapunov equation (9.6) may be unsolvable and formula (9.5) for the cost function may be false.

Secondly, to ensure optimal tracking it seems useful that the positive semi-definite matrices \( Q \) and \( V \) are invertible.

Thirdly, in the paper the optimization method for Problem 1 is presented somewhat differently using techniques from optimal output feedback control system design and a Lagrange multiplier approach.

Fourthly, substituting the solution (9.7) to Problem 1 into the cost function \( J(K, F) \) we obtain the criterion function

\[
J(K) = J(K, F) = \bar{r}^T V \bar{r} - \bar{r}^T V H A_K^{-1} B \left( B^T A_K^{-T} (P_K + H^T V H) A_K^{-1} B \right)^{-1} B^T A_K^{-1} H V \bar{r}.
\] (9.8)

Thus Problem 2 reduces to the problem of finding a feedback gain matrix \( K \) such that the closed loop state matrix \( A_K \) is asymptotically stable and

\[
\bar{r}^T V H A_K^{-1} B \left( B^T A_K^{-T} (P_K + H^T V H) A_K^{-1} B \right)^{-1} B^T A_K^{-T} H^T V \bar{r}
\]

is minimal.

Fifthly, it would be interesting to study the robustness of the solution given by the algorithm, for example, robustness with respect to wind disturbance.
10 Optimal output tracking problem by predictive control

Main reference: Singh-Steinberg-DiGirolamo [55].

10.1 Model of the aircraft

The type of aircraft is not specified. The model has been derived in Rhoads-Schuler [51] and Hacker-Oprisiu [24].

10.1.1 Modelling assumptions

In the paper it is assumed that the throttle control is used to keep the speed $V$ constant.

10.1.2 Modelling equations

10.1.2.1 Equations of motion

The equations of motion are given by the following system of nonlinear equations:

\[
\begin{align*}
\dot{p} &= \ell_{\beta}\beta + \ell_q q + \ell_r r + (\ell_{\beta\alpha}\beta + \ell_{\alpha r} r)\Delta\alpha + \ell_p p - i_1 qr + (\ell_{\alpha \Delta\alpha} + \ell_{\alpha A})\alpha_A + \ell_{\alpha R}, \\
\dot{q} &= \bar{m}_{\alpha}\Delta\alpha + \bar{m}_q q + i_2 pr - m_{\beta}\beta + m_{\Delta\alpha} (\ddot{\phi}) (\cos \theta \cos \phi - \cos \theta_0) + \bar{m}_{\beta E} \\
\dot{r} &= n_{\beta}\beta + n_r r + n_{q p} + n_{p a} p\Delta\alpha - i_3 pq + n_{\alpha A} + (n_{\delta A} + n_{\alpha A})\Delta A + y_{\delta R} \\
\dot{\alpha} &= q - p\beta + z_{\alpha}\Delta\alpha + (\ddot{\phi}) (\cos \theta \cos \phi - \cos \theta_0) + z_{\beta E}, \\
\dot{\beta} &= y_{\beta}\beta + p(\sin \alpha_0 + \Delta\alpha) - r \cos \alpha_0 + (\ddot{\phi}) \cos \theta \sin \phi + y_{\beta A} + y_{\beta R}, \\
\dot{\phi} &= p + q \tan \theta \sin \phi + r \tan \theta \cos \phi, \\
\dot{\theta} &= q \cos \phi - r \sin \phi,
\end{align*}
\]  

(10.1)

where

\[
\Delta\alpha = \alpha - \alpha_0. 
\]  

(10.2)

For the meaning of the symbols in (10.1), (10.2) is referred to Rhoads-Schuler [51] and Hacker-Oprisiu [24].

10.1.2.2 Aerodynamic equations

The aerodynamic equations are part of the system (10.1).

10.1.2.3 Actuator equations

The actuators are ideal.

10.1.2.4 Wind equations

Wind is not modelled.
10.1.2.5 Sensor equations

The sensors measure $p$, $q$, $r$, $\alpha$, $\beta$, $\phi$, and $\theta$. All sensors are ideal and measure continuously.

10.2 Mathematical structure of the model

10.2.1 Definitions

10.2.1.1 State vector for the model of the aircraft

The state vector $x$ is given by

$$x = (p \quad q \quad r \quad \alpha \quad \beta \quad \phi \quad \theta)^T.$$

10.2.1.2 Control inputs

The control inputs are the aileron deflection $\delta_A$, the rudder deflection $\delta_R$ and the elevator deflection $\delta_E$. Put $u = (\delta_A \quad \delta_R \quad \delta_E)^T$.

10.2.1.3 Exogenous inputs

Not applicable.

10.2.1.4 Measurement outputs

The measurement output $y$ is given by $y = x$.

10.2.1.5 Regulated outputs

The paper describes two models for the regulated output. The regulated output $z$ is given by $z_1 = z = (\alpha \quad \beta \quad \phi)^T$ for the first model and by $z_2 = z = (\theta \quad \beta \quad \phi)^T$ for the second model.

10.2.2 General form of the model

For both models the general form is

$$\dot{x} = f(x) + g(x)u,$$

$$y = x,$$

$$z = h(x),$$

(10.3)

where $x \in M \subseteq R^n$, $u \in U \subseteq R^m$, and $z \in R^m$. Notice that the number of inputs is equal to the number of outputs. The set $U$ is not specified. The set $M$ depends on the regulated output model. The complement of $M$ in $R^n$ has measure zero (in the paper it
is erroneously stated that \( M \) has measure zero) and this complement could be interpreted as a set of singularities. The precise definitions of \( M \) for the two regulated output models will be given later in Section 10.5.2.

### 10.3 Optimization problem

#### 10.3.1 Data

The data at time tag \( t \) are:

- for each model one output reference trajectory \( r(t), t_0 \leq t \leq t_f \),
- the state history \( x(t'), t_0 \leq t' \leq t \)

#### 10.3.2 Assumptions

The output reference trajectory \( r \) is generated by a command generator, which is specified in the paper.

#### 10.3.3 Statement of the optimization problem in words

The optimal output tracking optimization problem is to find a state feedback control law such that the regulated output of the closed loop system follows the reference trajectory as well as possible.

### 10.4 Mathematical formulation of optimization problem

#### 10.4.1 Criterion

To introduce the criterion for the approximate optimization problem, in the paper two assumptions are made on the nonlinear system (10.3). It is not explained why these assumptions are made. The first assumption is

- the system (10.3) is input-output feedback linearisable.

For the second assumption some notation needs to be introduced. For each \( 1 \leq i \leq m \), let \( \rho_i \) be the relative degree of the \( i \)-th output \( z_i \), i.e., \( \rho_i \) is equal to the smallest integer \( \rho \) for which \( z^{(\rho)} \) depends on the input \( u \) (if such an integer \( \rho \) does not exist, then \( \rho_i \) is defined to be \( \infty \)). Part of the second assumption is that the numbers \( \rho_i, i = 1, 2, \ldots, m \) are finite. Let \( Z \) be given by

\[
Z = (z_1^{(\rho_1)}, \ldots, z_m^{(\rho_m)}).
\]

Differentiating the output equation and substituting the state equation, one obtains that \( Z \) is of the form

\[
Z = a(x) + D(x)u.
\]
for some functions $a$ and $D$. The second assumption is

- the relative degrees are finite and the matrix $D(x)$ is invertible for any $x \in M$.

Let $s$ be the vector function $s = (s_1, s_2, \ldots, s_m)^T$, where for each $i = 1, 2, \ldots, m$ the function $s_i$ is a linear combination of derivatives and an integral of the error function $e_i$:

$$ s_i = e_i^{(\rho_i-1)} + k_{i,\rho_i-1} e_i^{(\rho_i-2)} + \ldots + k_{i,1} e_i + k_{i,0} \int_{t_0}^t e_i \, dt. $$

To define the criterion we introduce $\tilde{J}$ given by

$$ \tilde{J}(u(t)) = \tilde{s}(t + \Delta)^T Q \tilde{s}(t + \Delta) + u(t)^T R u(t), $$

where $\tilde{s}(t + \Delta)$ denotes the predicted value of $s$ at time $t + \Delta$. The prediction $\tilde{s}(t + \Delta)$ in (10.4) is taken to be the first-order Taylor polynomial

$$ \tilde{s}(t + \Delta) = s(t) + \Delta \dot{s}(t) = s(t) + \Delta (a(x(t)) + E(t) - r^{(\rho)}(t) + D(x(t))u(t)), $$

where $r^{(\rho)} = (r_1^{(\rho_1)}, \ldots, r_m^{(\rho_m)})^T$ and $E = (E_1, \ldots, E_m)^T$ with $E_i = q_i(\frac{d}{dt})(e_i)$ and $q_i(\lambda) = k_{i,\rho_i-1} \lambda^{\rho_i-1} + \ldots + k_{i,1} \lambda + k_{i,0}$, $i = 1, 2, \ldots, m$. Notice that $q_i(\frac{d}{dt})(e_i)$ depends on $x$ only for each $i$. The criterion $J(u(t))$ is defined to be the right hand side of (10.4) with the prediction $\tilde{s}(t + \Delta)$ replaced by the right hand side of (10.5).

10.4.2 Constraints

no constraints.

10.4.3 The approximate optimization problem: mathematical formulation

The approximate optimization problem is to find for each $t$ an input $u(t)$ which minimizes $J(u(t))$.

10.4.4 The approximate optimization problem: relation with the original optimization problem

For $R = 0$ one obtains perfect tracking.

10.4.5 Design parameters in the optimization problem

The design parameters are the gains $k_{i,0}, \ldots, k_{i,\rho_i-1}$, $i = 1, 2, \ldots, m$, the positive number $\Delta$, and the matrices $Q$ and $R$. The matrices $Q$ and $R$ are positive definite. The gains $k_{i,0}, \ldots, k_{i,\rho_i-1}$ are such that the polynomials $p_i$ given by

$$ p_i(\lambda) = \lambda^{\rho_i} + k_{i,\rho_i-1} \lambda^{\rho_i-1} + \ldots + k_{i,0}, \quad i = 1, 2, \ldots, m, $$

are Hurwitz, i.e., $p_i$ has all zeros in the open left half plane for each $i$. 
10.5 Optimization method or technique

10.5.1 Further assumptions

Recall that the following assumptions have been made:

- the system (10.3) is input-output feedback linearisable.
- the relative degrees are finite and the matrix $D(x)$ is invertible for any $x \in M$.

10.5.2 Description of the method or technique

The optimization problem has the following explicit solution

$$
    u(t) = -\Delta (R + \Delta^2 D(x(t))^T Q D(x(t)))^{-1} D(x(t))^T Q \\
    (s(t) + \Delta(a(x(t)) + E(t) - r^o(x(t))).
$$

(10.6)

Notice that both $s(t)$ and $E(t)$ in (10.6) are independent of $u$.

It is checked in the paper whether or not the aircraft modelling equations (10.3) satisfy the second assumption for Problem 1. Although the second component of $\dot{z}_1$ depends on $u$, according to the paper this dependability can be neglected, since the aileron, rudder, and elevator are principally moment producing devices. The first and third components of $\dot{z}_1$ are independent of $u$. The second derivative of $\dot{z}_1$ is of the form

$$
    \ddot{z}_1 = a_1(x) + D_1(x)u.
$$

The set $M = M_1$ in (10.3) is defined to be $M_1 = \{x \in R^n \mid \det D_1(x) \neq 0\}$. With this definition of $M$ the second assumption is satisfied. For the second model the set $M = M_2$ is defined in a similar way, neglecting the impact of the control input on the first two components of the derivative of $z$.

10.5.3 Further design parameters

no further design parameters.

10.5.4 Choice of the design parameters

The design parameters have been selected by trial and error.

10.6 Remarks

In the paper the following properties of the optimization method are shown:

- for nonzero but small values of $R$, the responses for $e_i$ are approximately decoupled,
- the error $e$ is bounded under certain admissible perturbations of the functions $f$ and $g$. The admissible perturbations are described in the paper.
10.7 Further references

In the paper Singh-Steinberg [54] the approach is worked out in more detail.

10.8 Comments

The second assumption "$D(x)$ is invertible for any $x \in M$" is equivalent to solvability of the strong input-output decoupling problem by regular state feedback (Theorem 8.9 in Nijmeijer-Van der Schaft [45]), since the system (10.3) is analytic. When the second assumption holds, it follows from Section 8.1 of Nijmeijer-Van der Schaft [45] that there exists a coordinate transformation such that the input-output relation of the transformed system is described by a linear system. Thus the second assumption implies the first assumption.
Part III
Conclusions and recommendations
11 Conclusions and recommendations

In this report an inventory on optimization problems, methods, and techniques has been presented. The inventory is part of the NLR contribution to GARTEUR Flight Mechanics, Systems and Integration Action Group FM(AG10) "Multivariable Optimization Techniques for Experimental and Conceptual Design". The optimization problems, methods, and techniques have been taken from the field of flight control design.

The inventory contributes to the development of an interface between mathematically formulated optimization problems from specific applications on the one side and multipurpose numerical optimization methods and techniques on the other side.

For the inventory a classification scheme has been developed to describe papers in flight control design that use optimization methods and techniques. Main elements in the classification scheme are

- models of aircraft, which are described both in physical variables and in their general mathematical form,
- optimization problems, which are described in terms of criterion functions and constraints,
- optimization methods and techniques, of which characteristics are given with special attention to the choice of design parameters.

From the inventory of optimization problems, methods and techniques in flight control design along the classification scheme it can be seen that:

- various models of aircraft can be found in the literature. Only a few generic models are needed to cover the models in the literature studied.

- three classes of optimization problems in flight control design can be found: optimal tracking problems, trajectory optimization problems, and optimal state estimation problems. Each of the three classes often has its own type of criterion functions: quadratic criterion functions in optimal tracking, nonlinear criterion functions in trajectory optimization, and quadratic (stochastic) criterion functions in optimal state estimation. Constraints on the variables in the model are taken into account explicitly in trajectory optimization problems.

- optimization methods and techniques may involve formulas for exact solutions and methods for non-exact solutions. Formulas for exact solutions appear mainly for solutions of optimal tracking problems and some optimal state estimation problems. Implementations of formulas for exact solutions may require numerical software to solve
certain types of equations, like differential equations or Riccati equations. Methods for non-exact solutions appear mainly for trajectory optimization problems and also for some optimal state estimation problems.

For the future work of GARTEUR Flight Mechanics, Systems and Integration Action Group FM(AG10) conclusions and recommendations can be made on

- a benchmark problem on optimization in flight control design,
- multi-objective optimization problems, methods, and techniques,
- the algorithms that should be available from the interface.

Optimization methods and techniques for trajectory optimization problems seem to involve the most difficult numerical problems among the three classes of optimization problems that have been identified. Thus the class of trajectory optimization problems is the most interesting from the point of view of the need for an interface between optimization problems and numerical algorithms. Therefore it is recommended to take a trajectory optimization problem as benchmark optimization problem. The associated criterion function should be nonlinear of the form

\[ J = S(x(t_f), u(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t) dt \]

and the optimization problem should include constraints.

In the literature on flight control design usually little attention is paid to the optimization methods and techniques that have been used. It is recommended to follow closely the developments in flight control design. A continuation of the present study could pay special attention to the optimization methods and techniques applied in Magni-Bennani-Terlouw [37] to the (multi-objective) optimal tracking problems and stability augmentation problems associated with the models RCAM and HIRM, that have been defined in the design challenge reports [1] and [2] (see the accompanying report [5]). Furthermore, it may be useful to carry out a study to see whether Zhao [65] is still up-to-date for optimization methods and techniques applied in trajectory optimization. Among the numerical algorithms which should be available by the interface, algorithms for estimation and filtering deserve special attention. Although such algorithms are used in flight control design, they are not included in numerical software like Matlab. The NAG Fortran Library contains only a few algorithms [8], Chapter G13. In addition, it is recommended to look for general numerical methods to solve problems in flight control design.

As a first step to the multi-objective interface it is recommended to carry out a study on multi-objective optimization. Such a study should not be restricted to multi-objective
optimization in flight control design, since the literature on multi-objective optimization in flight control design seems to be limited. Therefore it is recommended to extend the study to multi-disciplinary optimization and multi-objective optimization in general. From the non-flight-control optimization problems the literature study may reveal general principles that can also be applied for multi-objective optimization in flight control design. Furthermore, it is recommended to study the mathematical literature on multi-objective optimization, like simultaneous $H_2/H_\infty$-optimization.

The interface needs both numerical and exact algorithms. Hence it should contain numerical and computer algebra software. For numerical software it is useful to have several algorithms available for solving the same problem. In this case a decision tree is very helpful, see e.g. in Chapter E04 on "Minimizing or Maximizing a Function" in the introductory guide of the NAG Fortran Library [8]. When such a decision tree cannot be made easily, in other words, when it is not easy to say on beforehand which algorithm is the best, then it is useful to have several algorithms available as well.

The results of the present inventory and the accompanying studies [5] and [6] by GARTEUR FM(AG10) are to be used in a joint effort to define the interface, in a follow-on study.
References


